Statistical Lower Tolerance Limits for Rectangular Mortise and Tenon Joints

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Tests were conducted to determine the bending moment capacity of 215 red oak and 140 white oak T-shaped rectangular mortise and tenon joints. Rails measured 22.2 mm by 63.5 mm in cross section; tenons measured 32 mm in length by 38 mm in height by 9.5 mm in thickness. Specimens were assembled with a 40% solid content polyvinyl acetate adhesive. The average bending moment capacity of the red oak specimens was 353 Nm with a standard deviation of 48 Nm; in the white oak specimens, it was 358 Nm with a standard deviation of 62 Nm. The lower tolerance limits of the red oak specimens at the 75|75, 90|75, 75|90, 90|90, and 95|95 confidence|proportion levels were 318, 316, 289, 286, and 266 Nm, respectively, whereas in white oak specimens, the values were 314, 308, 273, 268, and 240 Nm, respectively. Overall, the results indicated that the use of statistical lower tolerance limits procedures provide a systematic means of relating standard deviations to mean values in determining reasonable design values for the moment capacity of the joints. Conclusions were not reached concerning which confidence|proportion level might be best suited for determining reasonable design values for furniture joints, but the results did illustrate the consequences of a given choice.

Keywords: Statistical lower tolerance limits; Rectangular mortise and tenon joints

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INTRODUCTION

The rational design of furniture frames dictates that the moment-resisting capacities of the members and joints of a frame are known, so that both can be designed to resist the internal forces imposed on them in service. Considerable information concerning joint and member capacity is available (Pang *et al.* 2011; Likos *et al.* 2012; Kasal *et al.* 2015; Podlena and Boruvka 2016), but reasonable design values for members and joints have not been established. Furthermore, the subject of allowable design values have, for the most part, not been well-addressed in the literature, although related studies do exist. For example, Eckelman (1974) suggested that the reasonable design stress for the front rails of sofas could be taken as one-third of the modulus of rupture, and Ratnasingam *et al.* (2010) stated that the allowable design capacity for cyclically loaded mortise and tenon joints is 20% of their ultimate static load capacity.

Given the paucity of information that exists on the subject, designers are left to rely on personal judgment and experience to select reasonable design levels for joints. In practice, a designer must estimate the expected range of values, as well as the average value, especially what fraction of the average value can be used for design purposes. A somewhat intuitive, rule-of-thumb practice is to set the design value for a specific joint configuration equal to the average capacity of the joint minus one corresponding standard deviation. The merit of this procedure lies in its simplicity. However, whether the results of tests carried out on a limited number of samples are truly representative of the characteristics of the joints is unclear.

These considerations lead to the conclusion that design capacities for joints should be expressed in terms that statistically define the expected performance of the joints. Therefore, statistical methods have been developed to analyze heterogeneous sources of data (Trindade and Uryasev 2006). One-sided statistical tolerance limits resolve such issues (Natrella 1963) because they enable a designer to estimate, to a specified degree of confidence, a value above which a proportion of test values should occur (Lee and Sa 2001). Such limits are determined by the expression $LTL = \bar{x} - k \times s$, where LTL refers to the lower tolerance limit, $\bar{x}$ refers to the average capacity of the test results, s refers to the standard deviation of the results, and k is a tolerance factor “such that the probability is γ that at least a proportion P of the distribution will be greater than $\bar{x} - k \times s$, based on a sample size of n” (Natrella 1963). K-factors are obtained from a number of published sources, including Natrella (1963) and Lieberman (1957). In addition, Link (1985) provided an expression to calculate k-factors.

When considering lower tolerance limits for a given set of joint data, the question of immediate concern is what fraction of the average value for the set is associated with each lower tolerance limit. Additionally, it is not known how many values of the data set fall below a chosen LTL, and how far they lie below it.

The answers to these questions are not obvious, but an LTL approach provides the flexibility to determine levels appropriate for design practice. Furthermore, the method is suitable for different structural needs. High confidence and proportion levels are needed for critical joints such as the side rail to the back post joints in chairs without stretchers, whereas lower levels are appropriate for joints in frames with multiple members, such as the side stretcher to front and back post joints in Shaker chairs. Overall, the selection of appropriate levels requires extensive testing of both laboratory and factory samples.

The initial questions that need to be answered when applying statistical tolerance limits to furniture joint design capacities are a) what are the appropriate confidence/proportion limits of furniture joints, and b) how do the average and standard deviations tend to approach their final values as the number of test samples increase. The purpose of this paper is to examine these two issues.

The objectives of this paper were as follows: (1) to obtain initial insight into appropriate confidence|proportion levels for one configuration of rectangular mortise and tenon furniture joints for 2 different wood species; (2) to obtain visual insight into the relationship between LTL and sample size; and (3) to examine the consequences of the selection of a given confidence|proportion level on design values.

EXPERIMENTAL

Plan of Study

The cumulative averages and standard deviations for sample sizes of 5 through n_1 (215) red oak (*Quercus rubra*) specimens and 5 through n_2 (140) white oak *(Quercus alba*) specimens by increments of 5 for joints of a single joint geometry were computed. Based on the average and standard deviation obtained for each sample size, the corresponding lower tolerance limits at five arbitrarily chosen confidence|proportion levels (in this case

75|75, 75|90, 90|75, 90|90, and 95|95) were determined and compared to the average for n_1 and n_2 specimens. Finally, the number and distribution of test values that fall below the confidence limits were evaluated.

Materials

Red oak and white oak wood, which are widely used in furniture construction, were selected for the tests, and 1-m long boards were obtained from a local sawmill/lumber dealer. Each of these boards was cut from the end of a full-length board at the mill. All 1-m samples were conditioned and maintained at 7% moisture content and subsequently machined to a thickness of 22.2 mm. Defect-free 63.5 mm wide by 305 mm long components were machined from the 1-m boards. All components were sequentially numbered.

The rail and post for each specimen were then randomly selected from the resulting material pool. Sufficient material was obtained to fabricate 215 red oak and 140 white oak specimens (Fig. 1).

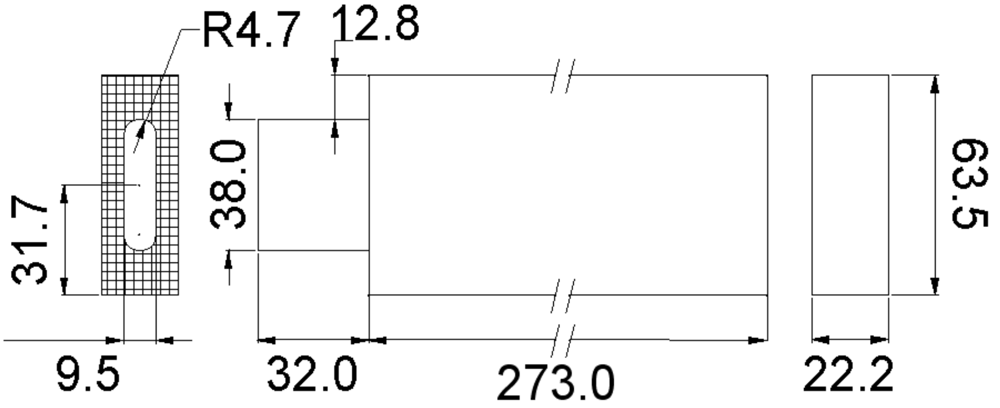

Fig. 1. Rail dimensions in mm; overall dimensions of rail and post (22.2 mm × 63.5 mm × 305 mm)

Tenons, 32 mm long by 38 mm wide by 9.5 mm thick, were cut with a tenoning machine. Matching mortises were cut with a router. Tolerances were such that a tenon could be inserted 2/3 of its length into a mortise without using under force.

The faces of the tenon and the walls of the mortise were coated with a 40% solid content polyvinyl acetate (PVA) adhesive (Franklin International, Columbus, USA), and the full length of the tenon was inserted into the mortise and clamped in place. Specimens remained clamped for 24 h, and they were stored in a conditioning room at 7% moisture content. At least one week elapsed before specimens were tested (Likos *et al.* 2013).

Test Procedure

All tests were conducted on an MTS universal testing machine (MTS Systems Corp. Minneapolis, USA) as shown in Fig. 2. The rate of loading was 12.7 mm/min. The moment arm was 254 mm. Loading continued until a non-recoverable drop-off in load occurred (Erdil *et al.* 2005)

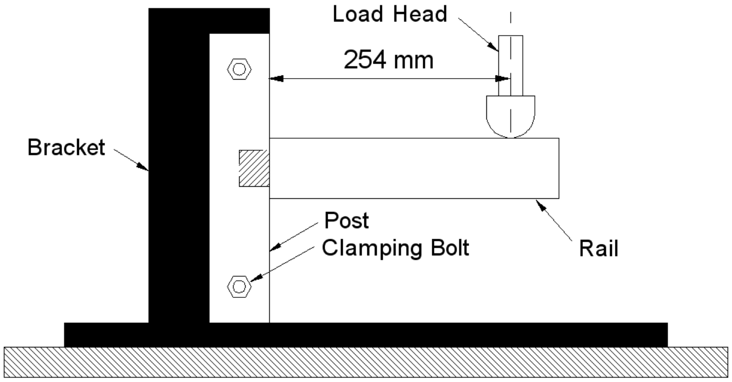

Fig. 2. Test set-up

RESULTS AND DISCUSSION

Red Oak Specimens

Means, standard deviations, highs, and lows

Results for the tests of the red oak specimens are illustrated in Fig. 3. The ultimate average bending moment capacity of the 215 red oak specimens (Fig. 3) was 353 Nm, with a standard deviation (STD) of 48 Nm, and the high and low test values were 464 Nm (131.4% of avg.) and 138 Nm (39.1% of avg.), respectively.

Cumulative average and standard deviation versus sample size

The manner in which the cumulative average and standard deviation changed as a function of the number of samples tested is illustrated in Fig. 3. The cumulative averages for 3, 5, 50, and 215 samples were 358 Nm, 339 Nm, 353 Nm, and 353 Nm, respectively. For this set of samples, the rounded cumulative average for 50 specimens (353 Nm) was identical to the rounded average of 215 specimens.

The standard deviation was somewhat less regular and ranged from a low of 32 Nm for 15 samples to a high of 59 Nm for 55 samples. The STD for a sample size of 50 was 51 Nm *versus* the final cumulative value of 48 Nm. The increase in STD for 55 specimens was largely due to the low capacity of the 52nd specimen at 138 Nm.

Confidence levels

The LTL values for the 75|75, 90|75, 75|90, 90|90, and 95|95 confidence|proportion levels for 215 red oak specimens were 318, 316, 289, 286, and 266 Nm (Fig. 3). The specimens at the 90|75 confidence|proportional level (316 Nm) differed little from those obtained at the 75|75 level (318 Nm). Likewise, the LTL for the 75|90 confidence| proportion level (289 Nm) differed little from the 90|90 LTL (286 Nm).

The cumulative 75|75 LTL values are also illustrated in Fig. 3. The cumulative 75|75 confidence|proportion levels for 3, 5, 50, and 215 specimens were 309, 284, 312, and 318 Nm, respectively. Overall, the LTL values varied from a low of 284 Nm for 5 specimens to a high of 323 Nm for 15 specimens. Also, the LTL increased in an essentially regular manner from the 2nd lowest value of 291 Nm for 20 specimens (resulting from low value of 140 Nm for the 19th specimen) to the final LTL of 318 Nm for 215 specimens.

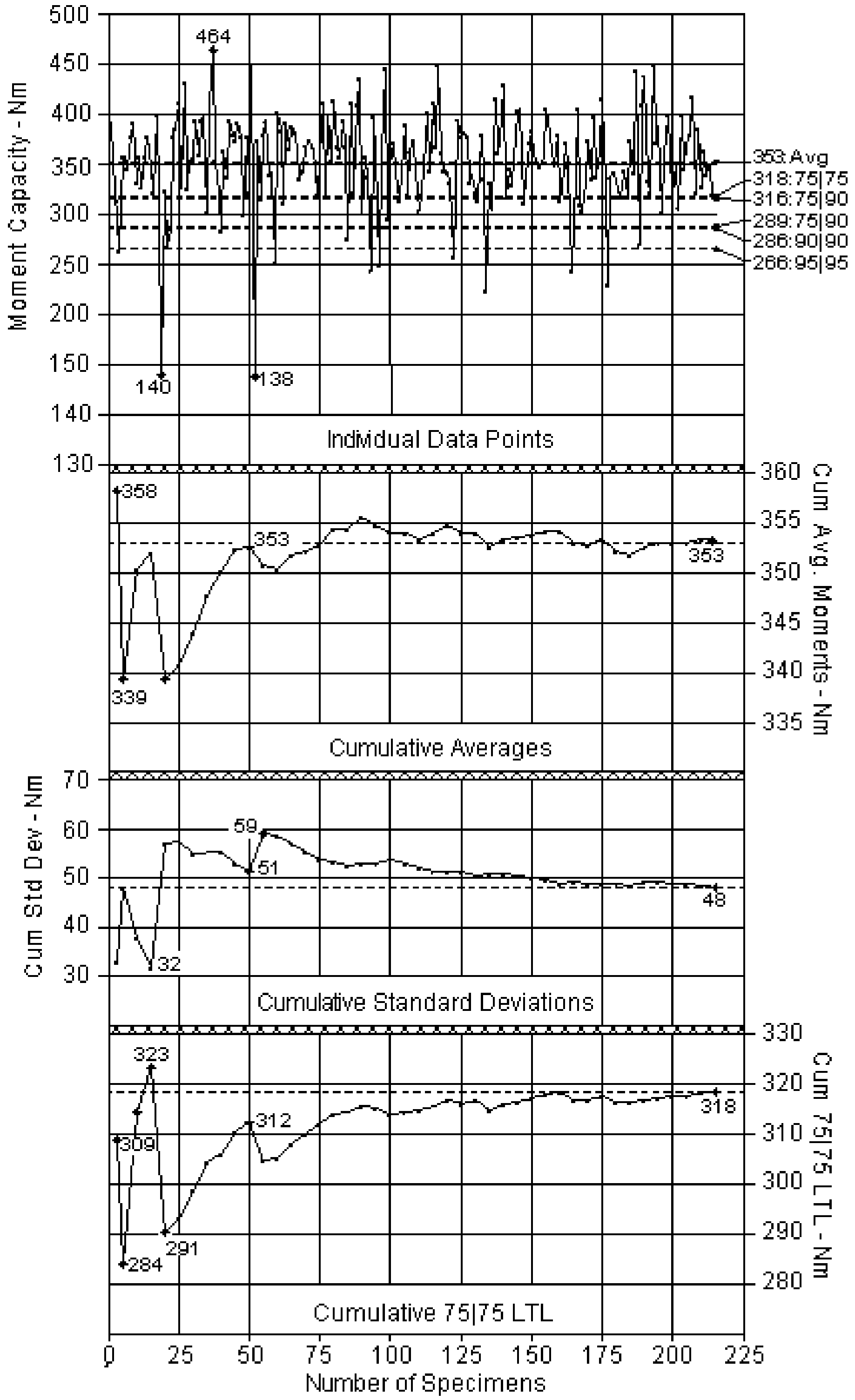

Fig. 3. Individual test results, cumulative average moments, standard deviations, and cumulative 75|75 LTLs for red oak specimens

Distribution of values below LTLs

At the 75|75 level, only 36 red oak specimens (16.7% of the total) had a lower capacity than the corresponding 318 Nm LTL. Similarly, the number of specimens that failed below their corresponding LTLs at the 90|75, 75|90, 90|90, and 95|95 levels were 32, 16, 15, and 10, respectively (Fig. 3).

The distributions of test values below corresponding designated LTL values are given in Table 1. At the 75|75 confidence|proportion level, of the 36 values that were less than their LTL, 21 values (58.3%) were 0 to 10% less, 7 (19.4%) were 10 to 20% less, 6 (16.7%) were 20 to 30% less, and 2 (5.6%) were 50 to 60% less than the LTL. Thus, the values that were less than the LTL still tended to cluster near it. This relationship also held true for the other confidence|proportion levels.

Table 1. Distribution of Test Values below Designated LTL Values: Number of Test Values below LTL within Specified Percentage Ranges

<table>
<tr><th rowspan="3"></th><th rowspan="3">Confidence | Proportion</th><th rowspan="3">LTL (Nm)</th><th rowspan="3">LTL (% of Avg.)</th><th rowspan="3">No. of Values Below LTL</th><th rowspan="3">No of % of Total</th><th colspan="6">No. of Values below LTLs</th></tr>
<tr><th colspan="6">Percent less than LTL by ranges</th></tr>
<tr><th>0-10</th><th>10-20</th><th>20-30</th><th>30-40</th><th>40-50</th><th>50-60</th></tr>
<tr><td rowspan="5">Red oak
n=215
Avg.=353.2</td><td>75|75</td><td>318.4</td><td>90.2</td><td>36</td><td>16.7</td><td>21</td><td>7</td><td>6</td><td>0</td><td>0</td><td>2</td></tr>
<tr><td>90|75</td><td>316.1</td><td>89.5</td><td>32</td><td>14.9</td><td>19</td><td>5</td><td>6</td><td>0</td><td>0</td><td>2</td></tr>
<tr><td>75|90</td><td>288.7</td><td>81.7</td><td>16</td><td>7.4</td><td>7</td><td>5</td><td>2</td><td>0</td><td>0</td><td>2</td></tr>
<tr><td>90|90</td><td>285.9</td><td>80.9</td><td>14</td><td>6.5</td><td>6</td><td>5</td><td>1</td><td>0</td><td>0</td><td>2</td></tr>
<tr><td>95|95</td><td>265.5</td><td>75.2</td><td>10</td><td>4.7</td><td>6</td><td>2</td><td>0</td><td>0</td><td>2</td><td>0</td></tr>
<tr><td rowspan="5">White oak
n= 140
Avg.=357.9</td><td>75|75</td><td>312</td><td>87.8</td><td>35</td><td>25.2</td><td>19</td><td>11</td><td>3</td><td>1</td><td>0</td><td>1</td></tr>
<tr><td>90|75</td><td>307.9</td><td>86</td><td>30</td><td>21.6</td><td>16</td><td>9</td><td>3</td><td>1</td><td>0</td><td>1</td></tr>
<tr><td>75|90</td><td>272.9</td><td>76.2</td><td>13</td><td>9.4</td><td>8</td><td>3</td><td>1</td><td>0</td><td>1</td><td>0</td></tr>
<tr><td>90|90</td><td>268</td><td>74.9</td><td>11</td><td>7.9</td><td>7</td><td>2</td><td>1</td><td>0</td><td>1</td><td>0</td></tr>
<tr><td>95|95</td><td>240.4</td><td>67.2</td><td>4</td><td>2.9</td><td>2</td><td>1</td><td>0</td><td>1</td><td>0</td><td>0</td></tr>
</table>

White Oak Specimens

Means, standard deviations, highs, and lows

Results from the white oak specimens are illustrated in Fig. 4. The ultimate bending moment capacity of the 140 white oak specimens was 358 Nm, with a standard deviation of 62 Nm. High and low test values were 476 Nm (132.9% of avg.) and 151 Nm (42.3% of avg.), respectively. It is interesting to note that the cumulative average for the white oak specimens was essentially the same as the red oak specimens' average, but with a 22.9% greater standard deviation.

Cumulative average and standard deviation versus sample size

The manner in which the cumulative average and standard deviation changed as a function of the number of samples tested is illustrated in Fig. 4. The cumulative averages for 3, 5, 50, and 215 samples were 379, 360, 367, and 358 Nm, respectively (Fig. 4). Thus, the cumulative average for 50 specimens (367 Nm) was 2.5% greater than the final average, which consisted of 140 specimens (358 Nm).

The results for standard deviation were somewhat less regular. The STDs for 3, 5, 50, and 140 specimens were 75, 59, 57, and 62 Nm, respectively. The lowest STD (47 Nm) occurred at a sample size of 10**.**

It is also noteworthy that, for sample sizes of 140, the average for red oak (353 Nm) differed little from white oak (358 Nm). However, the STD of 51 Nm in red oak was substantially less than that of 62 Nm in white oak.

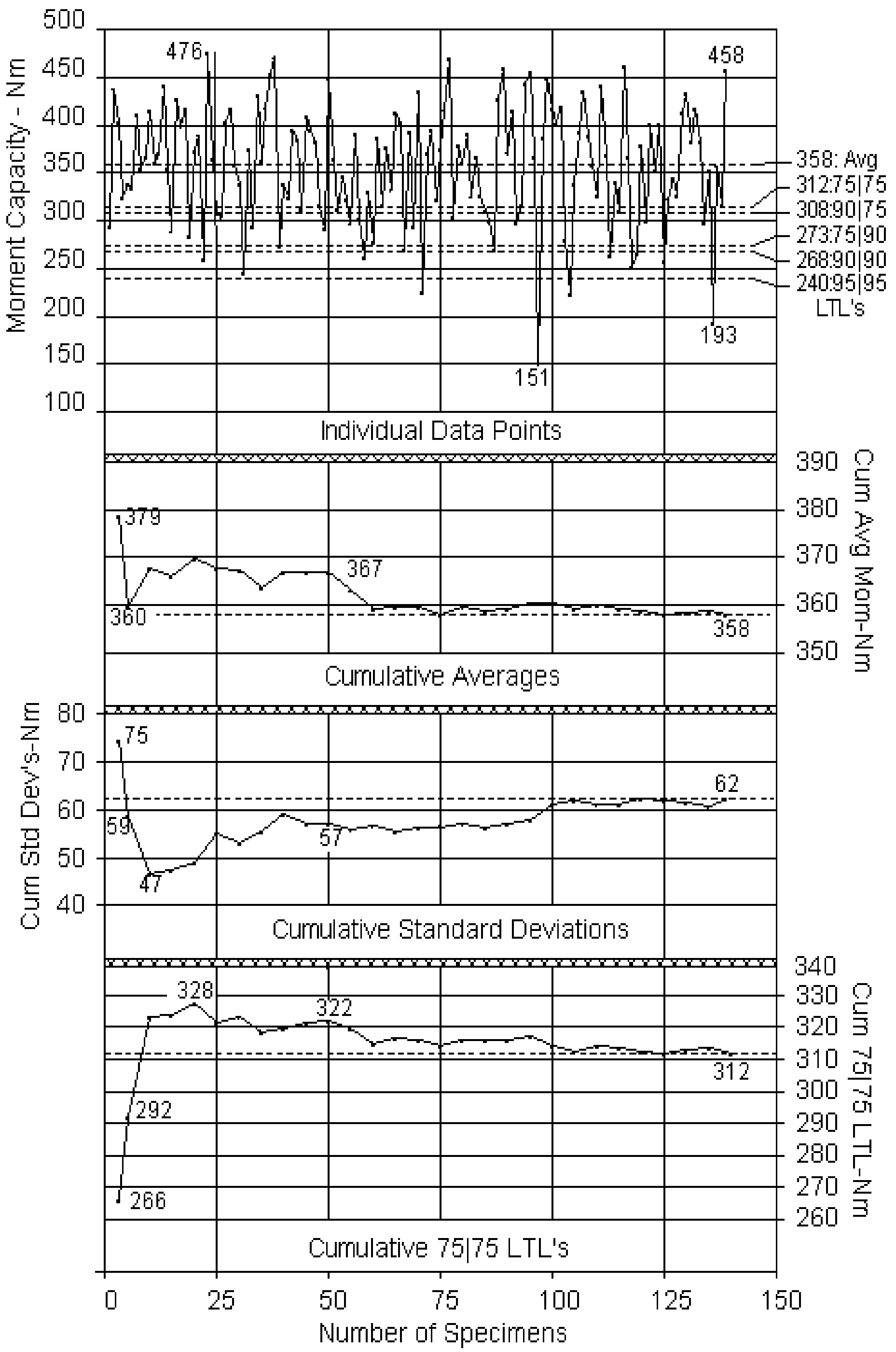

Fig. 4. Individual test results, cumulative average moments, standard deviations, and cumulative 75|75 LTLs for white oak specimens

Confidence levels

The LTLs for 75|75-, 90|75-, 75|90-, 90|90-, and 95|95 confidence|proportion levels for 140 white oak specimens were 312, 308, 273, 268, and 240 Nm, respectively (Fig. 4). The results at the 90|75 confidence proportional level (308 Nm) differed little from those at the 75|75 level (312 Nm). Likewise, the LTL for the 75|90 confidence|proportion level (273 Nm) differed little from the 90|90 LTL (268 Nm).

The cumulative 75|75 LTLs are also illustrated in Fig. 4. The cumulative 75|75 confidence proportion levels for 3, 5, 50, and 215 specimens were 266, 292, 322, and 312 Nm. Overall, the LTLs varied from a low of 266 Nm for 3 specimens to a high of 328 Nm for 20 specimens. Also, the LTLs decreased in an essentially regular manner from the high of 328 Nm for 20 specimens to the final cumulative value of 312 Nm for 140 specimens.

Distribution of values below LTLs

At the 75|75 level, only 35 white oak specimens (25.2% of the total) had less capacity than the corresponding LTL of 312 Nm. Similarly, the number of specimens that failed below their corresponding LTLs at the 90|75-, 75|90-, 90|90-, and 95|95 levels amounted to 30, 13, 11, and 4, respectively (Fig. 4).

The distributions of test values below corresponding designated LTL values are given in Table 1. At the 75|75 confidence|proportion level, of the 35 values that were less than the LTL, 19 values (54.3%) were 0 to 10% less, 11 (31.4%) were 10 to 20% less, 3 (8.6%) were 20 to 30% less, 1 (2.9%) was 30 to 40% less, and 1 (2.9%) was 50 to 60% less than the LTL. Thus, the values that were less than the LTL tended to be clustered adjacent to it. This relationship was the same for the other confidence|proportion levels.

CONCLUSIONS

1. The use of statistical lower tolerance limit procedures provides a systematic method of determining the implications of the use of specified fractions of the average capacity of a given joint for design purposes.
2. Overall, the results do not define the confidence|proportion levels that are best-suited for furniture. However, they do indicate the consequences resulting from the selection of a specific level, and how it determines what fraction of the average value might be used as a “reasonable” design value.
3. The results did not provide definitive answers to the question of how many specimens should be tested to determine reliable lower tolerance limits for a given joint configuration. However, the results do illustrate how the average and standard deviation tend to approach the final values as the number of test samples increases. The results also illustrated the effect of the occurrence of “weak” specimens in the average and standard deviation as the number of samples increased.
4. Consideration of the concepts involved in this study illustrates the means for relating the results of tests to standard deviations, rather than to simple averages, which is particularly important in comparing the performance (reliability) of different types of joints. Thus, a systematic consideration of the standard deviations associated with various joint types can aid a designer in selecting a joint best suited for critical

applications. For example, within a given joint type, consideration of the standard deviations associated with given configurations may lead to definitions of “preferred” joint geometries.

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