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ON THE DISTRIBUTION OF PORE HEIGHTS IN RANDOM LAYERED FIBRE NETWORKS

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INTRODUCTION

At a recent research meeting [1], discussion of measurements of internal pore heights from cross sections of paper by K. Niskanen and H. Löytty and of surface topography by M. Lorusso highlighted the need for an appropriate reference model. This short communication fills a gap in the existing theory, which dates back to the work of Corte and Kallmes [2,3]; it provides the analytic distribution for internal and surface pore heights in a layered random fibre network. Such a network is known to resemble laboratory made paper from dilute suspensions of fibres.

The importance of the development lies in the influence of surface pore structure on printing and fluid entry and the influence of internal void structure on compressibility and hence also printing and converting of paper. We have already a good understanding of the statistical features of horizontal pore structure in paper; now we have a model for the vertical component in random paper. It turns out that the standard deviation of pore height is approximately proportional to the mean pore height.

PORE HEIGHT STATISTICS

Two types of pores arise in a random fibre network consisting of n similar layers:

Surface pores: a surface pore is a sequence of adjacent voids in successive layers, bounded in the network by a fibre; so we have a free surface then a vertical sequence of r voids followed by a fibre.

Interior pores: an internal pore is a succession of voids bounded at each end by a fibre; that is, a fibre then a vertical sequence of r voids, then a fibre.

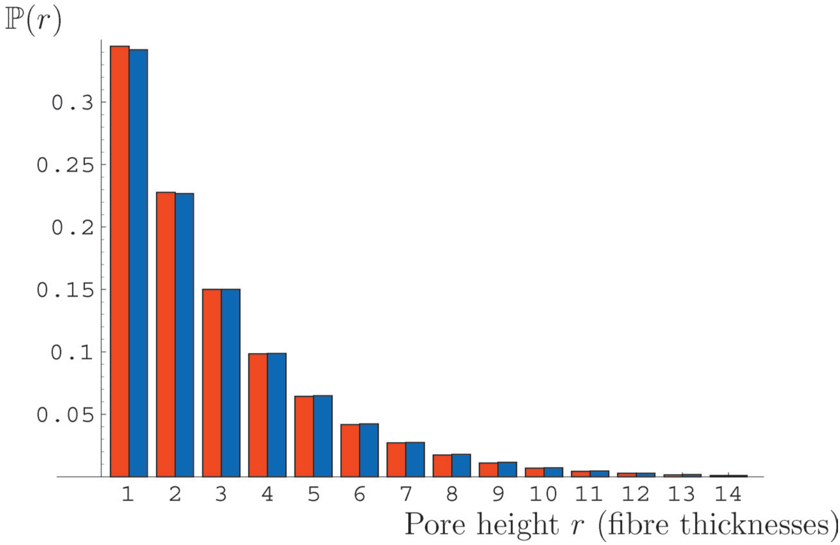

Figure 1 Probability distribution of internal pore heights (left columns) and surface pore heights (right columns) in units of one fibre thickness for a random sample of $n = 20$ layers, each layer having mean solid fraction $p = 0.3$. Both have mean pore height about 1.8 units and standard deviation about 2.2 units.

Now, in a vertical section through a random fibre network, the vertical sequence of fibres and voids is a stochastic process subordinate to the binomial distribution, since at each layer level a fibre is either present with probability p, the solid fraction, or absent with probability $1 - p$. Then in a stack of n layers the mean number of fibres is np and its variance is $np(1 - p)$, but it is not immediately clear what will be the distribution of heights of sequences of voids. Evidently the distribution of such heights r, measured in units of one fibre thickness (here equivalent to the thickness of one layer), also is controlled by the binomial distribution.

Recall that the relationship between the mean coverage c in a random fibre layer and the mean solid fraction p in the layer is

$$(1 - p) = e^{-c}. \quad (1)$$

So, for example, in a layer with $p = 0.3$, the mean coverage is $c \approx 0.36$, only 5% of the area has more than 1 fibre coverage and less than 0.6% has more than 2 fibre coverage. The net grammage in a stack of $n = 20$ such layers is typically in the range 40–50 gm^{-2}; it is given by np times the mean fibre grammage. Analytic expressions for the two distributions of pore height are given in the Appendix. It turns out that both distributions are similar, as may be seen for the examples shown in Figure 1, for a network of $n = 20$ layers with each layer having mean solid fraction $p = 0.3$. The mean pore height is about 1.8 fibre thicknesses, with standard deviation about 2.2. The two main variables are the number of layers n, and the solid fraction p in a layer; their effects on the statistics of the distribution of pore heights are illustrated in Figure 2 and Figure 3, respectively. These results seem in agreement with

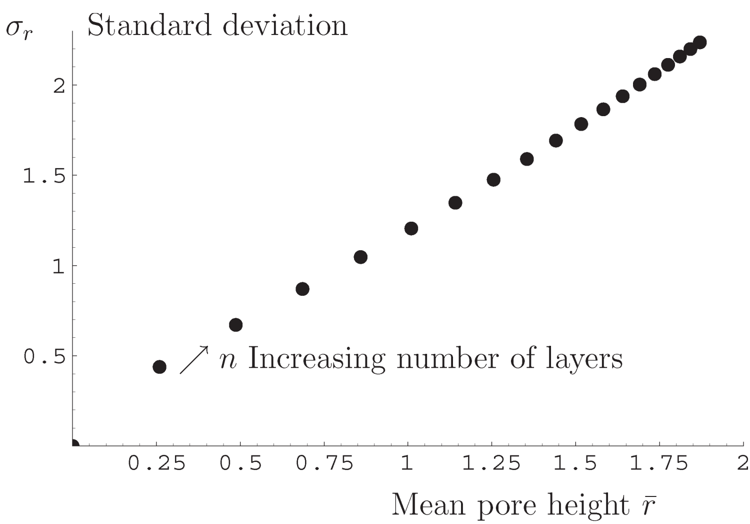

Figure 2 Effect of number of layers n. Plot of standard deviation σ_r against mean $\bar{r}$ for pore heights in units of one fibre thickness for random samples of up to $n = 20$ layers, each layer having mean solid fraction $p = 0.3$. The standard deviation is for many practical purposes proportional to the mean. Both internal and surface pores have similar plots; mean and standard deviation increase monotonically with increasing n.

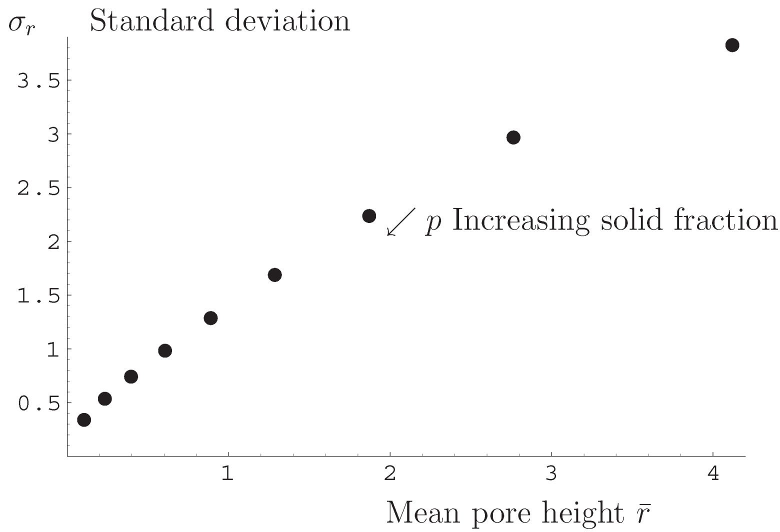

Figure 3 Effect of layer solid fraction p. Plot of standard deviation σ_r against mean $\bar{r}$ for pore heights in units of one fibre thickness for random samples of 20 layers, each layer having mean solid fraction $0.1 \leq p \leq 0.9$. The standard deviation is for many practical purposes proportional to the mean. Both internal and surface pores have similar plots; mean and standard deviation decrease monotonically with increasing p.

experimental data from cross sections; this and extension of the analysis to non-random networks will be pursued elsewhere.

The interesting point is that for pore height, like many other geometrical features of random networks, here again we find that the standard deviation is for many practical purposes proportional to the mean. For horizontal pore sizes, such a proportionality persists even for *non-random*, flocculated structures; the derivation of the pore height distribution in non-random networks is an open problem for the future.

APPENDIX

Surface pores

Consider a stack of n Poisson layers, each with solid fraction p. The probability distribution function $\mathbb{P}_{Surf}\,(r,p,n)$ for **surface pore height** r and its mean $\bar{r}_{Surf}$ are obtained from the binomial distribution and given after some algebraic simplification by

$$\mathbb{P}_{Surf}(r,p,n) = (p(1-p)^r(n-r-1))\,/\left(\sum_{r=0}^{n-1} p(1-p)^r(n-r-1)\right),$$

$$= \frac{(1-p)^r p^2 (n-r-1)}{-1+(1-p)^n + np}, \qquad n = 0, 1, \ldots, n-1.$$

$$\bar{r}_{Surf} = n - 2 + \frac{2}{p} - \frac{(n-1)np}{-1+(1-p)^n + np}$$

$$\text{for } n = 0, 1, \ldots, n-1.$$

Internal pores

The corresponding probability distribution function $\mathbb{P}_{Int}(r,p,n)$ for **internal pore height** r and its mean $\bar{r}_{Int}$ are obtained similarly and reduce to the following

$$\mathbb{P}_{Int}(r,p,n) = (p^2(1-p)^r(n-r-2))\,/\sum_{r=0}^{n-2}(p^2(1-p)^r(n-r-2)),$$

$$= \frac{(1-p)^{1+r} p^2 (n-r-2)}{-1+(1-p)^n + p(n+p-np)}, \qquad n = 0, 1, \ldots, n-2.$$

$$\bar{r}_{Int} = \frac{((1-p)^n(2+(-3+n)p)) + (-1+p)^2(-2+(-1+n)p)}{p(-1+(1-p)^n + p(n+p-np))}$$

$$\text{for } n = 0, 1, \ldots, n-2.$$

The standard deviations σ_{Surf} and σ_{Int} also are known analytically and available from the author, but their expressions are somewhat cumbersome so they are omitted here.

ACKNOWLEDGEMENT

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Transcription of Discussion

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A prepared contribution was also given by Rune Holmstad from NTNU/PFI as a back up to this work, this follows the discussion on this paper

Byron Jordan Paprican

I would like to compliment you on the elegance of your straightforward mathematical demonstration. I think there are two areas which you need to extend it to make it a bit closer to the real world. One is that if I can calculate from your model that I have a pore that is 4 fibres deep, in order to know how deep that is in microns, I am going to have to also do a four-fold Laplace convolution of the fibre thickness in order to get the true distribution. The other thing is that in some, but not most, cases we are interested in pores that are not vertical, and so my question is how would one extend this to consider pores that have real tortuosity.

Kit Dodson

Yes, I've always found that it's very easy to invent work for people to do. Luckily Rune and Marielle Lorusso did take up with the suggestion to make measurements. Your point is well taken. There are two things that perhaps can be said. First, we know the planar pore size distribution, the horizontal, for random and non-random fibre networks, and the convolution of the vertical and horizontal pore structure is a theoretical development that is in hand – Bill Sampson and I will report on that at another time. It doesn't have the clinical clarity of this because the functions become less desirable before coffee! But just touching on reality, I can report that Marielle Lorusso measured the effect of the mean on the standard deviation and it fits.