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STIMULATING MATHEMATICS IN THE PAPER INDUSTRY

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It is a surprise and an honour to be asked to deliver this address. It gives me an excellent opportunity to make some remarks about the gulf that exists between mathematicians and other researchers, as the following story illustrates. A mathematician was about to deliver a prestigious lecture at an Oxford College. The Master, a geographer, asked me to give him a joke that he could use to put everyone in a relaxed mood. I suggested the old one "There are three kinds of mathematician: those that can count and those that cannot". The Master took the podium and announced that he had a joke to start off the proceedings. After due reflection he began "There are two . . . ." But he went no further.

This joke anticipates the main message of my talk which is how to bridge the gulf between mathematics and the real world. I will begin with an "over the top" example, then focus on what seems to me to be the really important dialogue between mathematics and the paper industry, and finally conclude with some general remarks and pointers for the future.

Take a piece of paper or a towel with a wet patch in the middle and try to dry it by a very direct but very impractical method, namely just sucking the moisture out from a point somewhere in the middle (e.g. with a hypodermic syringe). Then, if you look sideways on, you see a curve which is a very crude representation of the boundary of the wet region; of course life is much more complicated but even this simplistic scenario can lead you to something very delicate mathematically. If you are an expert on porous medium flow you might model the flow of the moisture by Laplace's equation for the pressure. You would have to put some boundary conditions on the equation on the boundary of the moisture region and you would have to impose some condition at the sink. It's quite easy to prove that this mathematical problem is a very 'nice' problem if you are *injecting* moisture into a source instead of *withdrawing* it to a sink; the 'bounding' curve will then expand and will turn into a circle, but if you extract the moisture, it is very easy to prove that this curve is wildly unstable and can do some very strange things.

It is with great hesitation that I will try a little experiment here. I expect that you know that there is a strong analogy between a porous media flow and what is called "Hele-Shaw" flow between parallel plates. I have a drop of washing-up liquid which I will now put on a transparency. So could you think of the bounding curve of the moisture region as the boundary of the drop when I put another transparency on top of the first. When I press down with the second transparency, it's analogous to moisturising the original sheet of paper; when I lift it off it is analogous to drying. You can see that as I push the transparencies together the drop expands and has become circular, and now when I pull the transparencies apart you can see the instability. So even if you don't believe the mathematical analogy, I have just proved it for you in practice!

Now let me tell you how this leads you into one of the most fascinating mathematical investigations that I have ever carried out, not that it has much practical relevance. The crucial step is to go into the complex domain, and instead of thinking of the paper in the x-y plane, think of it in the z-plane, $z = x + iy$. We want to find the bounding curve, i.e. the moisture front, and the way mathematicians would think of that most conveniently is to think of a function f of z, that also depends on time (but time is just a parameter) which takes the bounding curve into a circle in the f-plane. If we could find this function f we would know exactly where the bounding curve was. Now it is obvious physically that, if this function is going to make sense, every point in the moist region in the physical plane must map to a unique point in the f-plane and vice versa. This is called "one-to-one" mathematically; for each z there is a unique f and for each f a unique z, and, when you think about it, most mapping functions are not like that. If f was z then obviously every point z would go to every point f and vice versa, but, if f was z^2, every time you went around the origin once in the z-plane you would go around twice in f so the mapping couldn't possibly be "one to one". You might ask "suppose f was nearly z, say $z = f + \alpha f^2$?" Well it is very easy to show that, if you let α start off from zero, the unit circle in the f-plane will initially correspond to a what's called a limaçon, but when $\alpha = 0.5$ the corresponding curve has a cusp, and is called a cardioid. If you go beyond $\alpha = 0.5$, the mapping gets so far away from z that the corresponding curve turns over on itself, so the mapping doesn't make physical sense any more. So, if you think of the wetting problem in terms of finding f, you have to satisfy a strong condition; mathematically it is called "univalency" and this is where the model ties in with something more dramatic theoretically. Back between the wars, there was a very famous mathematical conjecture called the Beiberbach Conjecture, which used to be almost on a par with Fermat's last theorem and the Reimann Hypothesis. This conjecture was proved in the 1960's; as you know, Fermat's theorem was proved more recently and the Reimann Hypothesis remains unproved. Now Fermat's theorem has absolutely no practical applications whatever that I am familiar with, but, from what I have said, the Beiberbach Conjecture has at least tenuous relevance to the paper industry. It is much easier to state than the Reimann Hypothesis and it concerns the univalency of a function written as an infinite power series $z + a_2 z^2 + a_3 z^3 + \ldots$.. I showed you the case when there was just an $a_{2,}$ and things went wrong when a_2 was a 0.5. The Beiberbabach Conjecture is that as long as a_n is less that 1/n in modulus for all n, the function is univalent, but it took a two hundred page paper to eventually prove it. We still don't know a lot about various details; in my example, when f became univalent, the bounding curve just developed a cusp but there are examples when it becomes fractal. This loss of univalency or "extreme univalency" may one day be related to the very complicated shape of my detergent drop!

So now let me spend the rest of the time talking about something much more down to earth, and I am going to do this chronologically and personally. I have been involved with "maths-in-industry" workshops for 30 years now. I haven't time to tell you all the ins and outs of their organisation, but I have selected some of the paper industry problems that I have encountered, and I hope you will be interested to see the way we have made an interface between mathematicians and paper researchers at these workshops. Let us go back to 1969 when there was a company called British Insulated Calendar Cables. They made paper for insulating electrical machinery, cables and wires, and the paper had to have the right electrical properties, so efficient drying was essential., [Ref. 1]. The mathematicians from Oxford and from outside gathered together and saw that BICC had made a quite crude model of the moisture content per unit volume, and he had an argument that it satisfied a nonlinear diffusion equation, which is a generalisation of Laplace's equation including a time derivative. Moreover, he made the diffusion coefficient nonlinear, and it so happened that his equation, which we had never seen before in 1969, was starting to become popular in the world of mathematics. It is called the "porous medium equation", and we now know that it is much more interesting than the usual heat conduction equation. But at the time we couldn't understand a lot of what BICC said. Indeed, when I dug out the report from our workshop, I discovered that the mathematicians wrote at the end of the week "the Faculty participants wanted more experimental data on the macroscopic properties of layered paper before having any confidence in the model proposed by BICC". What appalling arrogance! However it was a case of the biter bit, because, had we followed this model up, we would have been in at the ground floor of a large interesting mathematical investigation into the porous medium equation. As it was we trailed in the footsteps of many other mathematicians around the world. So here was a case where interaction between mathematics and the paper industry could have been fantastically beneficial to mathematics, but it was spoiled by a naive and uninquisitive attitude.

Now let's go on a little further to a piece of research that had a much happier ending [2]. It was written by my ex-supervisor, who was stimulated by a problem brought by another company, Formica, who wanted to impregnate paper with resin. The academic output was that Alan Tayler was able to write a paper in a learned journal about the parameter regimes for pressurised impregnation of paper using rollers. The paper comes between the rollers with a thickness s, and it is being compressed to thickness h, the purpose of the compression being to squeeze as much resin as possible into the paper. After some very delicate and very nice mathematics, Alan found a condition on the compression ratio h/s for the nip not to "block up". Formica certainly liked this formula; it was an example of mathematics and the paper industry acting together in a really mutual beneficial way.

My next example does not really concern paper, but I mention it from the public relations point of view. In 1980, when I was an editor of probably the best applied mathematics journal in the world [3], I received a paper from an Australian colleague whose work was motivated by towel drying in the laundry industry, but it is basically the same problem as hanging up a piece of paper to dry. He did a very comprehensive mathematical and computational analysis. He wrote the heat and mass transfer equations in proper "conservation-law" form, including evaporation fronts, like the moisture front that I told you about right at the beginning. He thus developed a very nice code that will handle the drying of laundry. I tell you this because, a few weeks after this paper was published, I received a call from an editor of *Scientific American*, asking "how is it that mathematics can possibly have anything to do with laundry?". He was a physical scientist but he just did not believe that there could ever be an interface between mathematics and the problem of drying a towel. It is impossible to explain such a scientific phenomenon over the telephone of course, and this illustrates again this gulf that I was mentioning in my preamble.

Now lets just get a little bit more up-to-date. These workshops that we originally held in Oxford, are now held around the world, and in particular there have been some exceptionally successful ones in Vancouver where they have a "Pacific Institute of Mathematical Sciences". They had their first industry workshop in 1997 and this involved MacMillan Bloedel [4,5]. Their first problem concerned optimal plank production from cylindrical tree trunks. This leads to a really nice optimisation problem where you have to minimise the wastage knowing the different kinds of demands for different sizes of planks. This is like many problems in the clothing industry (e.g. optimal pattern cutting) and indeed there are many industrial problems of this sort where completely new optimisation theories need to be brought to bear. At the same workshop, MacMillan Bloedel also brought a more traditional problem about the stress in the thermo-roll. Again there is a nip, through which the paper passes and you have to find if the hot rollers can crack as a result of the thermal stress at the nip, which is quite cold compared to the core. This leads to a nice problem in thermoelasticity around the nip. Finally I noticed that just as I was leaving the office on Friday that a version of the problem Alan Tayler addressed was also brought to the United States industry workshop a couple of years ago [6] and a much more sophisticated analysis was carried out more on the lines of the "Ilic theory".

I hope there are enough examples here to show you the kind of activity that is, in my opinion, at the heart of the interface between maths and the paper industry.

I would like to conclude by just pointing out that the proliferation of mathematical modelling is not peculiar to the paper industry. It is part of a revolution that I see around the world in applied mathematics, whereby even governments are starting to realise that mathematics underpins a vast number of industrial processes. Indeed two governments have actually given support to mathematicians working in industry on a large scale. In Germany, they have institutes called "Frauenhöfer Institutes", which are funded to the tune of several million marks each year on the condition that the academics work on industrial problems and generate industrial income comparable to public support. In this country recently we have done a very similar sort of thing with "Faraday Institutes", and I am a scientific director of such an Institute. We have been in existence for one year, and I thought I would close by showing you some of the collaborations that have been set up in the last year. This dramatically illustrates the new recognition that so many industries really only get to grips with their problems if they understand their processes quantitatively by use of mathematics as and when appropriate.

- The prediction of <u>electromagnetic compatibility</u> as more and more complicated electronics come into play.
- The <u>Food Industry</u> is a vast source of problems for mathematicians. The modelling of scraped surface heat exchangers is just one example, which is relevant to paper making.
- <u>Mathematical Finance</u> is a wildly popular subject with students and is crucial as far as many of educational programmes are concerned.

- Risk, and especially risk from the weather, is very important and requires new methodologies such as time series and stochastic analysis.
- Optical Fibre Communications may be unpopular at the moment, but it still offers vast mathematical challenges.
- Radio Spectrum Allocation has stimulated much new "discrete mathematics".
- Medicine, for example tissue engineering is perhaps the fastest-growing and potentially the largest sector of our Faraday research.

In conclusion, I can make one concrete offer to people who don't know the sympathetic applied mathematicians in their home country . Because I operate with so many colleagues around the world I can probably put you in rapid contact with the right sort of person. Meanwhile, I am looking forward to coming and learning about some more mathematical problems during this week.

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