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STATISTICAL GEOMETRY OF A FIBROUS NETWORK

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Synopsis

The statistical geometry of fibrous networks is described in terms of the fibre and sheet dimensions and geometric probability. The method has been developed for random two-dimensional structures and extended to cover deviations from randomness (orientation and flocculation). It is also applied to a multi-planar structure as a first approximation to three-dimensional structures. Further approximations to three-dimensional networks are discussed. Experimental results for two-dimensional structures are presented.

La géométrie statistique d'un réseau de fibres

La géométrie statistique des réseaux fibreux est décrite en termes des dimensions des fibres et de la feuille et des probabilités géométriques. La méthode est développée pour des structures bi-dimensionnelles organisées au hazard et étendue pour couvrir les deviations des cas dûs au hazard (orientation, flocculation). On l'applique aussi à l'étude d'une structure multi-planaire en tant que première approximation d'une structure à trois dimensions. On discute aussi d'autres approximations pour des réseaux tri-dimensionnels. Des résultats expérimentaux concernant des structures bi-dimensionnelles sont présentés.

Statistische Geometrie von Fasernetzwerken

Die statistische Geometrie von Fasernetzwerken wird mit Hilfe der Faser- und Blattdimensionen und der geometrischen Wahrscheinlichkeit beschrieben. Das Verfahren wurde zunächst für zufällige *zweidimensionale Strukturen entwickelt, aber dann so erweitert, dass es auch zur Beschreibung der Abweichungen von der Zufälligkeit (Orientierung und Flockung) benutzt werden kann. Ebenso lässt es sich zur Beschreibung multiplanarer Strukturen als erste Näherung für dreidimensionale Netzwerke verwenden. Andere Näherungen für dreidimensionale Strukturen werden diskutiert und experimentelle Ergebnisse für zweidimensionale Strukturen werden mitgeteilt.*

Introduction

LOOKING upon the papermaking process from a general standpoint, the papermaker's activities can be summarised in a few sentences. He finds cellulose fibres in plants (or in fabrics), where they are arranged to serve the purposes of nature (or the cotton manufacturer). He destroys the original ordered structure of the fibres by separating them partially or completely from each other. He brings them together again in a new structural arrangement to suit his own purposes. Along the way, he may take steps to remove undesirable substances from the fibres before using them or he may treat them mechanically to alter their dimensions or physical properties. From a fundamental point of view, operations like these, although of great importance to the quality of the finished product, are only incidental to the process. *Dissolution of an existing structure and creation of a new one is the essence of papermaking.*

The two outstanding characteristics of the structure produced on a papermachine or in a laboratory sheetmachine are disorder and planar shape. In either process, fibres are deposited on the wire in more or less haphazard fashion, but, through a combination of the hydrodynamics of the wire and the pressures applied in pressing and drying, they are forced into planes essentially parallel to that of the sheet.

It is generally accepted that, apart from the effects of additives, fillers, etc., the physical properties of the finished paper are determined primarily by the properties of the fibres and their arrangement in the sheet—that is, its structure. Through this symposium, we hope to obtain a better quantitative understanding—and this is the only true form of understanding—of the nature of sheet structure, because it is both the determinant of the physical properties of paper and the issue of the papermaker's effort.

Definition of the problem

THE structure of paper is defined here as the geometric arrangement of fibres and interfibre spaces in the sheet. The problem is to describe structure in terms of the shape of the fibres and geometric laws alone. The results ought to be generally valid for any fibre network, provided it satisfies certain conditions. *Not* included in these conditions are the structural features of the individual fibres. We will consider fibres as structural elements, characterised only by their dimensions and shape, a large number of which comprise an irregular network. Questions of how the network was produced or how it would respond to changes in its environment are immaterial. Our sole objective here is to describe the network correctly in its final, static condition. The problem being the geometry of an irregular system of a large number of elements, the only possible method is statistical geometry. The problem will be considered solved when a number of *dependent variables* (that is, well-defined, geometric properties of the network apparently related to the macroscopic properties of paper) can be calculated from other *independent variables* (quantities that are given or chosen at will). The number of fibre/fibre crossings and the number of spaces in the network are typical dependent variables, whereas the length and width of the fibres are examples of independent variables.

The long-range goal we keep before us is to provide a set of numbers such that someone who may never have seen a piece of paper can still form a reasonable image of it from our quantitative description. It is a goal to be reached in steps (if at all) and perhaps by a path similar to the following.

General plan

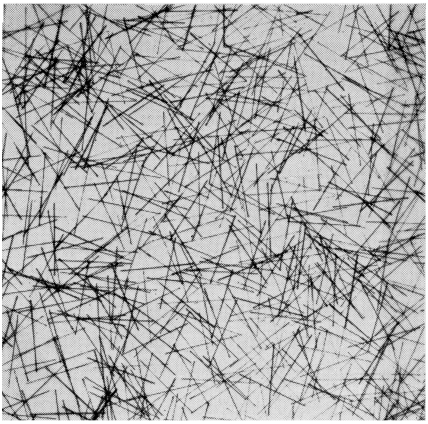
I. TWO-DIMENSIONAL (2-D) NETWORKS

- *A.* Ideal (random) case
- *B.* Deviations from randomness—
  - *1.* Non-random fibre orientation
  - *2.* Flocculation

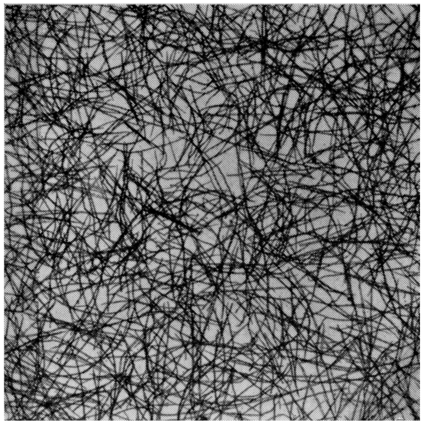
II. THREE-DIMENSIONAL (3-D) NETWORKS
(*three models have been considered*)

- *A.* Multi-planar networks
- *B.* Networks of horizontal fibres
- *C.* Squeezed-out three-dimensional random networks

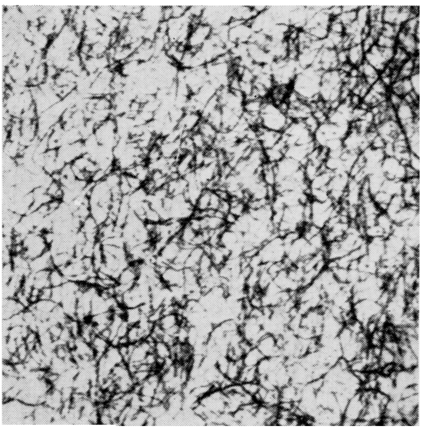
THE programme began with the two-dimensional case, because (*1*) the fibres of most papers arrange themselves in a series of parallel planes and (*2*) we wished to test our theories on very thin handsheets in which the fibres lie essentially in one plane and are clearly visible (Fig. 1*b*). A further simplification consisted in defining the two-dimensional (2-D) network to be ideally random—that is, the fibre centres and the angles between the fibres and a given direction have random distributions.



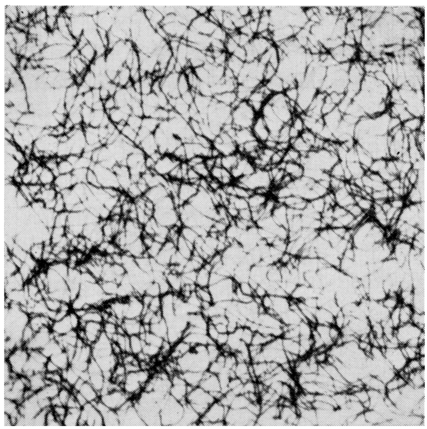
(a)—Random network of straight lines



(b)—Random 2-D sheet



(c)—Oriented 2-D sheet



(d)—Flocculated 2-D sheet

Fig. 1—2-D fibre networks

Real sheets differ from ideal 2-D structures not only by *not* being two-dimensional, but also by deviating from randomness. Deviations from randomness can be of only two types in that (*1*) the angular distribution is non-random (*orientation*) and (*2*) the fibre centres are not randomly distributed (*flocculation*).

Three-dimensional (3-D) structures are most easily obtained by piling up 2-D layers to form a sheet approximating real paper. Because drainage on papermachines takes place above the table rolls in steps, the fibres may be deposited stepwise on the wire, at least at low machine speeds. Therefore, we believe the multi-layer structure consisting of a pile of 2-D sheets to be a model worth considering, at least as a first approximation. A more realistic structure would be one in which the fibres are arranged in unevenly spaced, parallel planes. Still more realistic would be a 3-D fibre network squeezed out until the angles between the fibres and the plane of the sheet have all become comparatively small. If this last model could be made to include flocculation and non-random orientation, it would lead to the most realistic description of the structure of paper.

It is obvious from the foregoing general plan that the number of independent variables necessary to describe the structure increases with each succeeding entry. Near the bottom of the chart, our difficulties increased exponentially. Whenever they were more mathematical than conceptual, approximations were used; in other cases, they were such that no adequate definitions and concepts could be formed, leaving no alternative but to define artificial models. On the whole, we found that statistical geometry furnishes the tools for a satisfactory initial approach to the problem at hand.

The remainder of this article is given to relatively detailed considerations of the single steps of the general plan.

Ideal 2-D network

Definitions

IN an earlier publication giving full details,[1] an ideal 2-D network was defined to have certain properties—

1. The position of the fibre centres in the plane and angles between the fibre axes and a fixed direction have random distributions.

2. The area covered by more than two fibres is a negligible part (less than 1 per cent) of the total area.

3. Furthermore, fibres have length λ and width ω, both of which may vary so long as $\lambda \gg \omega$. Curvature in a fibre is expressed by the curl factor τ, the ratio of fibre length to the distance between its ends.[2] The values λ, ω and τ are independent variables (because they are fixed by nature and the processes leading to the finished structure).

Two other independent variables pertaining to the network as a whole are A, the area of the sheet and N_f, the total number of fibres in A. The quantities A and N_f may be replaced by w, the weight per unit fibre length (specific fibre weight) and W, the weight per unit area of sheet (basis weight of the sheet). Thus, we have—

$$N_f/A = W/\bar{\lambda}\bar{w} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(Throughout this article, the bars over symbols indicate mean values.)

Properties of random structures

FOLLOWING the definition of randomness above, the probability $p(r)$ that a point chosen at random in A is covered by r fibres is—

$$p(r) = \frac{e^{-N_f\bar{\alpha}/A}(N_f\bar{\alpha}/A)^r}{r!} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\bar{\alpha} = \bar{\lambda}\bar{\omega}$ is the mean area of a fibre and $N_f\bar{\alpha}/A$ is the mean number of fibres covering a point in A. The term $N_f\bar{\alpha}/A$ can be replaced by $W\bar{\omega}/\bar{w}$, according to equation (*1*). Then, with $p(r)_{r>2} < 0{\cdot}01$ (definition *2*), the maximum basis weight compatible with the definiton of a 2-D network can be calculated. For a number of papermaking fibres, this value has been shown to lie between 1 and 3 g/m^2 (*see* Table IV[1]).

An equation of the same form as (*2*) expresses the probability of finding exactly N_f fibre centres within a small area a, chosen at random in A. In this case, $N_f\bar{\alpha}/A$ is replaced by the mean number of fibre centres in the area a and $\bar{n}_f = N_f a/A = Wa/\bar{\lambda}\bar{w}$. Dividing A into a large number of squares, each of area a, the fraction containing exactly n_f fibre centres, $p(n_f)$, is given by—

$$p(n_f) = \frac{e^{-N_f a/A}(N_f a/A)^{n_f}}{n_f!} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

If the squares of area a have sides of the same order of magnitude as the fibre dimensions, most fibres will pass through more than one square. Consequently, each one will contain *segments* of fibres that pass through (fibre chords), terminate in (fibre ends) or lie entirely within the square. The fraction of the all the squares containing n_{seg} fibre segments is given by—

$$p(n_{\text{seg}}) = \frac{e^{-\bar{n}_{\text{seg}}}(\bar{n}_{\text{seg}})^{n_{\text{seg}}}}{n_{\text{seg}}!} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2b)$$

where $\bar{n}_{\text{seg}}$ is the mean number of segments per square.

A straight line of length L (a scanning line) drawn through a random 2-D fibre network will intersect a certain number of fibre axes N. Dividing the scanning line into small lengths l, the probability of finding exactly n

intersections in l (or the fraction of sections containing n intersections) is given by—

$$p(n) = \frac{e^{-Nl/L}\,(Nl/L)^n}{n!} \qquad \ldots\ldots\ldots \quad (2c)$$

Here, Nl/L is the mean number of fibres intersecting length l and is independent of the direction of scanning, because of the random fibre orientation.

Equations (*2–2c*) are all applications of the Poisson distribution, a function describing the relative frequencies of random events, processes and arrangements. Agreement between experiments and values predicted by any one of the equations verifies the random fibre centre distribution in a general way. The most convenient, though not very sensitive comparison of theory with experiment is that between equation (*2c*) and the measured scanning distributions. When applied in different directions, scanning also provides a simple means of checking the randomness of the angular distribution of the fibres.

Scanning furnishes data for still another test of randomness of the network. Consider the distances (gap sizes) along the scanning line between two consecutive intersections with fibres. For a random or Poisson process, the probability that this distance has a value between g and $(g+dg)$ is given by the *negative exponential distribution*—

$$r(g) = \frac{N}{L}\cdot\exp\left(-\frac{N}{L}g\right)dg \qquad \ldots\ldots \quad (3)$$

where N/L is the mean number of intersections per unit length of scanning line or the reciprocal mean gap size.

Calculation of dependent variables

Definitions

THOSE dependent variables chosen to describe the structure of an ideal 2-D network are[(1)]—

1. The (total) length or mass of fibrous material (M), m;
2. The (total) number of fibre crossings (N_c), n_c;
3. The (mean) number of crossings per fibre $(\bar{c})$, c;
4. The (mean) free fibre length or gap size along a fibre $(\bar{g})$, g;
5. The (mean) number of free fibre lengths per fibre $(\bar{n}_g)$, n_g;
6. The (total) number of polygons or holes (N_h), n_h;
7. The (mean) number of sides per polygon $(\bar{n}_s)$, n_s;
8. The (mean) polygon area $(\bar{a}_h)$, a_h.

The symbols in parentheses at the end of each of the eight definitions above refer to net values (totals or averages) of the variables. We believe variables *2–5* to be associated with mechanical properties of the network and *6–8* with porosity properties. Firstly, they, then their distributions, will be considered.

Net values

The total length and mass in a 2-D sheet of area A have received the same denotation, M—

$$M\begin{cases} = N_f\bar{\lambda} \\ \\ = AW/\bar{w} \end{cases} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{length} \qquad \begin{matrix} = N_f\bar{\lambda}\bar{w} \\ \\ = AW \end{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{mass} \qquad \begin{matrix} (4) \\ \\ (4a) \end{matrix}$$

2-D sheets consist of intersecting fibres and the spaces between them. In networks of straight lines, these spaces are polygons, whose average number of sides $\bar{n}_s$ has been shown to be 4 when the lines are of either infinite[3] or finite length.[1] The fibres of real paper are themselves slightly curled, but the distances along fibres between adjacent crossings—the free fibre lengths or gaps—are small enough to be treated as straight lines bordering polygons.

Of the remaining six quantities, the number of crossings is first in importance, because the other five can be calculated from it. It is expressed by—

$$N_c = \frac{(N_f\bar{\lambda})^2}{\pi A\bar{\tau}^2}, \qquad (5)$$

from which it follows that the mean number of crossings per fibre is—

$$\bar{c} = \frac{2N_c}{N_f} \qquad (6)$$

and the mean free fibre length is—

$$\bar{g} = \frac{\bar{\lambda}}{\bar{c}\bar{\tau}}. \qquad (7)$$

The average polygon, having four sides, is formed by four crossings, each of which belongs to four separate polygons. Thus, in a network of *infinitely long lines of no width*, the number of polygons is equal to the number of crossings, whereas networks of *finite lines of no width* have fewer polygons in proportion to the number of fibres. In networks of *finite fibres with width*, a fraction of the number of polygons equal to the fractional covered area of the sheet is lost. The final expression for the number of polygons in a network is—

$$N_h = (N_c - N_f)\exp\left(-\frac{N_f\bar{\alpha}}{A}\right) \qquad (8)$$

The mean polygon area, however, is not affected by fibre width, because the narrowing of larger polygons owing to fibre width is exactly compensated by the complete covering and disappearance of smaller holes, hence—

$$\bar{a}_h = \frac{A}{N_c - N_f} \qquad \ldots\ldots\ldots \quad (9)$$

Finally, the average number of free fibre lengths per fibre is given by—

$$\bar{n}_g = \bar{c}\exp\left(-\frac{N_f\bar{\alpha}}{A}\right) \qquad \ldots\ldots\ldots \quad (10)$$

where $c \gg 1$ and the exponential term corrects for the same effect as in equation (*8*).

Distributions

An important feature of random fibre networks is non-uniformity. The network of Fig. 1*a*, consisting of 970 straight lines of uniform length, was constructed using random number tables. Fig. 1*b* pictures about 970 fibres in a portion of a 2-D handsheet photographed to the same scale. Comparison with Fig. 1*a* shows a remarkable similarity and emphasises the fact that, in spite of efforts by the papermaker to the contrary, paper must always be non-uniform in structure.

By viewing paper as a random structure, we ascribe to it an inherent non-uniformity, which no papermaker can hope to remove without actually controlling the deposition of individual fibres. At the same time, we open the door to treatment of paper structure and its *inherent non-uniformity* by statistical geometry to express the distributions of the dependent variables in terms of the independent variables.

The dependent variables possess either areal or frequency distributions. No. *1*, *2* and *6** have the former, for, when the sheet is divided into small squares of area *a*, the squares exhibit distributions with respect to these three variables. These distributions cannot be predicted exactly from theory, but they may be approximated by a technique based on equation (*2b*). Knowing (*a*) the frequency of squares containing different numbers of segments (equation *2b*) and (*b*) the relationship within a square between the number of segments and any one dependent variable such as number of crossings, substitution of the latter (*b*) in the former (*a*) yields an approximate areal distribution of the number of crossings per square.

* The distribution of the number of polygons was not considered, because many polygons lie in two or more adjacent squares

It has been shown[1] that the mean of the distribution $\bar{n}_{seg}$ given by equation (*2b*) is related to the mean number of fibres per square $\bar{n}_f$ by—

$$\bar{n}_{seg} = \bar{n}_f \bar{k}/\bar{\tau}^2 \qquad (11)$$

where

$$\bar{k} = \frac{4\bar{\lambda}}{\pi l}+1 \qquad (12)$$

and $l=\sqrt{\bar{a}}$.[4]

The relationships between dependent variables No. *1* and *2* and $\bar{n}_{seg}$ are—

$$m = n_{seg}\,\bar{\lambda}_{seg} = n_{seg}\,\bar{\lambda}_{seg}\,\bar{w} \qquad (13)$$

and

$$n_c = \tfrac{1}{2}\,\bar{p}\,n_{seg}\,(n_{seg}-1) \qquad (14)$$

where $\bar{\lambda}_{seg}$ is the mean segment length. The term $n_{seg}(n_{seg}-1)/2$ is the maximum number of crossings possible among n_{seg} segments and $\bar{p}$ is the probability that two segments intersect. From earlier data[1]—

$$\bar{p} = \frac{\pi \bar{n}_f \bar{\tau}^2(\bar{k}-1)^2}{8\bar{k}(\bar{n}_f \bar{k}-\bar{\tau}^2)} \qquad (15)$$

which, when $\bar{\lambda}\rightarrow\infty$ and $\bar{\tau}=1$ reduces to $\pi/8$, Deltheil's value for a random network of infinitely long fibres.[5]

Variables No. *3*, *4*, *5*, *7* and *8* refer to individual geometric elements, therefore have frequency distributions. The distribution of the number of fibre crossings per fibre is simply the scanning distribution equation (*2c*) with n replaced by c and Nl/L by $\bar{c}=2N_c/N_f$. If individual values of c are converted into values of n_g, the number of free fibre lengths per fibre, with equation (*10*), the c distribution can be transformed into the n_g distribution. Both distributions can be corrected for variations in the fibre length by summing over all fibre lengths for each value of c and n_g. The fibre length distribution is an independent variable that has to be given in terms of an empirical relationship or in the form of a histogram.

The distribution of the free fibre length is identical with the negative exponential distribution, equation (*3*), where N/L is related to the independent variables by—

$$\frac{N}{L} = \frac{2\bar{\lambda}N_f}{\pi A} \qquad (16)$$

An approximate distribution of the polygon area can be calculated from equation (*3*) by assuming that the polygon area is proportional to the square of the side length, the free fibre length—

$$a = \gamma g^2 \qquad (17)$$

and substituting equation (*17*) into equation (*3*). If, as is usually the case, the free fibre length is small compared to total fibre length—

$$\gamma \approx 2/\pi$$

as found by Goudsmit for random networks of infinitely long lines.[3]

Experimental programme and conclusions

THE independent variables λ, ω, τ, A and W are subject to straightforward measurement. On the other hand, N_f must be calculated from the scanning equation (*16*), w from equations (*16*) and (*1*). The necessary scanning data are most easily acquired by projecting the sheet on to a screen and counting the number of intersections N that fibres make with a straight line of length L or with an optical scanner.

Handsheets of 2·5 g/m^2 fulfilling definitions *2* and *3* (page 17) were made from suspensions of 7×10^{-4} per cent consistency in a British sheetmachine, and studied in two ways—

(*a*) Results of scanning tests on them were compared directly with the distributions from equations (*2c*) and (*3*)—*see* Fig. 2 and 3, respectively.

(*b*) Portions of the sheet were projected on to a screen with a grid whose squares corresponded to 1 mm squares on the sheet. The number of segments and the number of fibre centres in each square were counted and their distributions over the sheet compared with curves predicted by equations (*2b*) and (*2a*)—Fig. 4 and 5, respectively.

The agreement in Fig. 2–5 between the calculated curves and the experimental results is strong evidence for the random nature of 2-D sheets and justifies the use of these equations for other areal and frequency distributions.

Fig. 6–8 demonstrate that the distributions of m, n_c and a_h can be calculated with a remarkable degree of accuracy from the independent variables. Now, it is not surprising that the net values showed good agreement between theory and experiment.[1]

That equations derived for geometric properties of 2-D sheets agree with experiment implies *that an ideal 2-D fibre network is completely described when we assume the sheetforming process to be random and assign values to its independent variables.*

Orientation in a 2-D network

So far, we have been concerned with the geometric properties of randomly formed networks only. In this section, we will consider the way in which arbitrary angular distributions of fibre axes influence the geometric properties

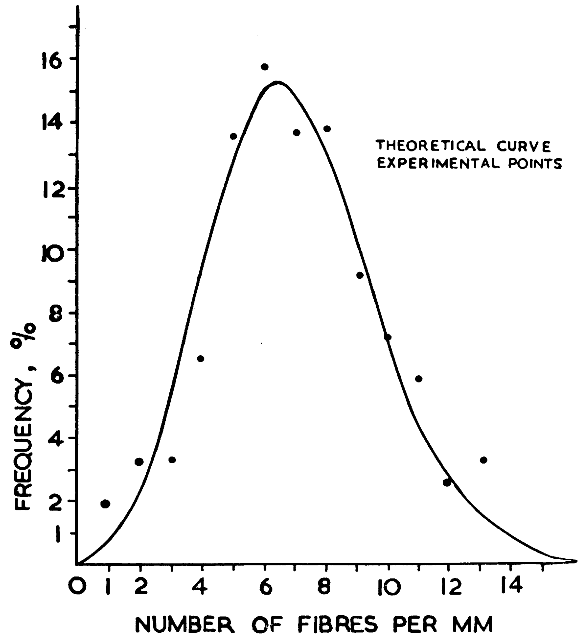


Fig. 2—Distribution of the number of fibres intersecting a scanning line

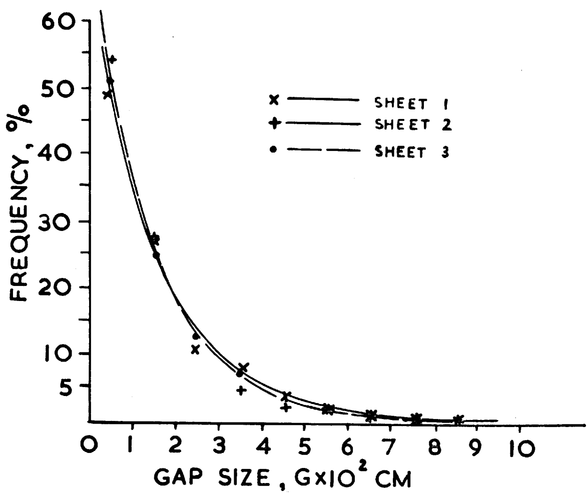


Fig. 3—Distribution of free fibre length

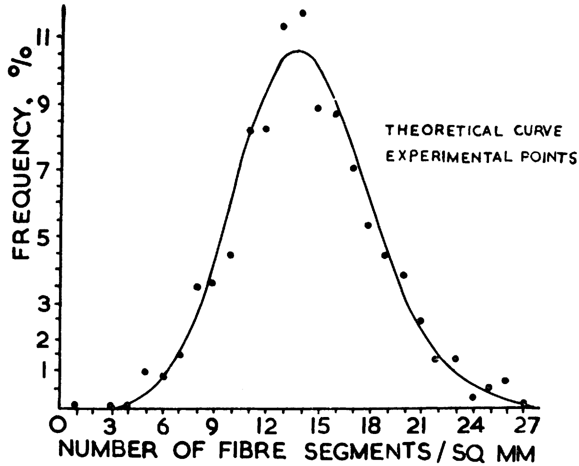


Fig. 4—Distribution of the number of fibre segments per mm²

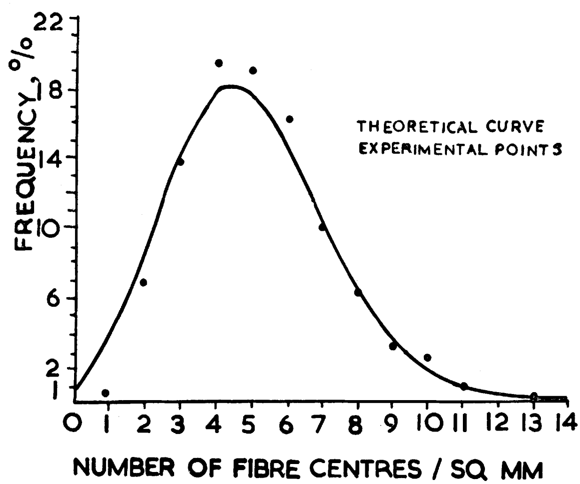


Fig. 5—Distribution of the number of fibre centres per mm²

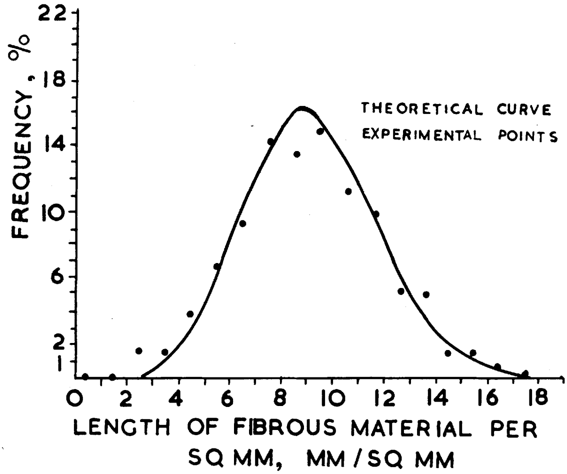


Fig. 6—Distribution of the length of fibrous material per mm²

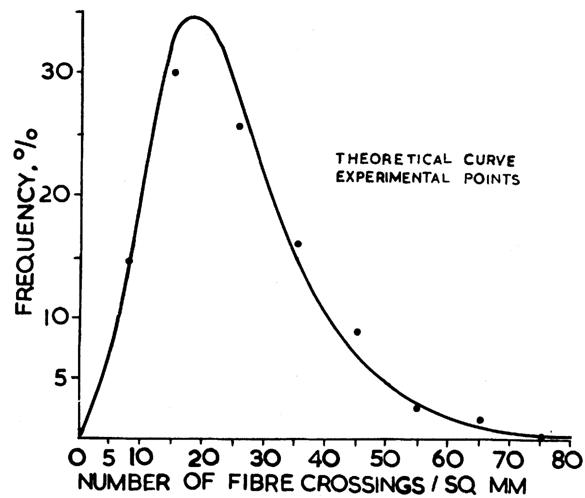


Fig. 7—Distribution of the number of fibre crossings per mm²

of 2-D sheets. Because the dependent variables can best be calculated from N_c, we derive this property first, starting with a generalised form of equation (*14*)—

$$N_c = \tfrac{1}{2}PN_f(N_f-1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14a)$$

The term P is the probability of an intersection between two given straight lines of lengths λ_i and λ_j, lying in the area A and making angles θ_i and θ_j with a fixed direction. Numerically—

$$P = \frac{\lambda_i\lambda_j \sin \mid \theta_i - \theta_j \mid}{A} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19a)$$

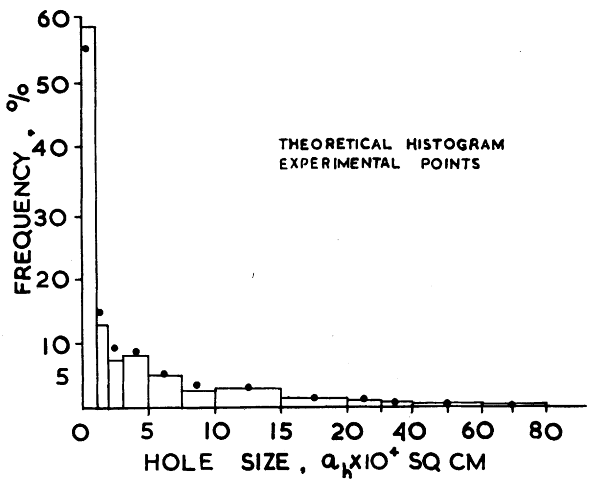


Fig. 8—Distribution of polygon area

Introducing the fibre length distribution $\Lambda(\lambda)$ and the angular distribution $\Theta(\theta)$ and averaging over all fibres, one obtains—

$$P = \frac{1}{A}\int_{\theta_i=0}^{\theta_i=\pi}\int_{\theta_j=0}^{\theta_j=\pi}\int_{\lambda_i=0}^{\lambda_i=\infty}\int_{\lambda_j=0}^{\lambda_j=\infty} \lambda_i\lambda_j \sin|\theta_i-\theta_j|\,\Theta(\theta_i)\Theta(\theta_j)\Lambda(\lambda_i)\Lambda(\lambda_j)\, d\theta_i d\theta_j d\lambda_i d\lambda_j$$

$$= \frac{\bar{\lambda}^2}{A}\int_{\theta_i=0}^{\theta_i=\pi}\int_{\theta_j=0}^{\theta_j=\pi} \sin|\theta_i-\theta_j|\,\Theta(\theta_i)\Theta(\theta_j)\, d\theta_i d\theta_j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

Solution of equation (*19*) requires evaluation of $\Theta(\theta)$. Rather than measure the angles that the individual fibres make with a given axis (a cumbersome job), we employed the scanning technique to measure $\Theta(\theta)$. In oriented fibre networks and machine-made papers, the number of intersections with a scanning line of length L in the machine-direction, $N(0)$, $(\theta=0)$, will be smaller than the number in the perpendicular (cross-) direction, $N(\pi/2)$. Scanning such sheets in different directions defines a periodic function $N(\theta)$, which can be represented by a Fourier series; the first two terms suffice for an initial approximation—

$$N(\theta) = a+b\cos 2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The double angle is used for symmetry reasons; a and b are constants.

It can be shown that the angular fibre distribution $\Theta(\theta)$ is related to equation (*20*)[4]—

$$\Theta(\theta) = \frac{1}{\pi}+e\cos 2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where e is the eccentricity of the distribution (*see* Appendix 1)—

$$e = -\frac{3b}{\pi a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The machine-direction ($\theta=0$) must be found experimentally, if it cannot be inferred, as in oriented handsheets. One scans the network in different directions radially from an arbitrarily chosen origin. The method of least squares gives expressions for a, b and the negative angle α that the machine-direction makes with the initially chosen origin as follows—

$$a = \frac{\sum N(\theta_i)}{n}; \qquad b = \sqrt{m^2+p^2}; \qquad \alpha = \tfrac{1}{2}\tan^{-1}(-p/m) \quad . \quad (23)$$

where

$$m = \frac{\sum N(\theta_i)\cos 2\theta_i}{\sum\cos^2 2\theta_i} \quad \text{and} \quad p = \frac{\sum N(\theta)\sin 2\theta_i}{\sum\sin^2 2\theta_i}$$

Once the machine-direction is known, e is calculable also from the scanning results—

$$e = \frac{3}{\pi}\,\frac{N(\pi/2)-N(0)}{N(\pi/2)+N(0)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The resultant eccentricity is a new independent variable related to fibre orientation.

Substituting equation (*21*) into (*19*) eventually gives—

$$P = \frac{2\bar{\lambda}^2}{A}\left(\frac{1}{\pi}-\frac{e^2\pi}{6}\right).$$

Term P, equation (*14a*) and the reduced mean fibre length $\bar{\lambda}/\bar{\tau}=\bar{\lambda}_{i,j}$ combine to form—

$$N_c = \frac{N_f{}^2\bar{\lambda}^2}{A\bar{\tau}^2}\left(\frac{1}{\pi}-\frac{e^2\pi}{6}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The number of crossings is greatest when $e=0$, that is, when the network is random, at which equation (*25*) reduces to equation (*5*).

Although N_c is diminished by any preferential fibre orientation, the relationships between N_c and the other dependent variables are unaffected. Thus, a decrease in N_c increases the mean polygon area [equation (*9*)] and the mean free fibre length [equation (*7*)] and decreases the total number of polygons [equation (*8*)], the number of crossings per fibre [equation (*6*)] and the mean free fibre length [equation (*10*)]. Of course, the number of crossings and mean free fibre length of a given fibre is a function of its orientation. This leads to the result that when a strip of paper is clamped in machine-direction the clamp will grip more fibres, but each fibre will, on the average, be crossed by fewer other fibres. Similarly, when clamped in the cross-direction, the clamp will grip fewer fibres, but each fibre will, on the average, be crossed by more other fibres. The ratio of total number crossings made by the fibres gripped when the sheet is clamped in the machine-direction $S(\pi/2)$ and the corresponding number for the cross-direction $S(0)$ is given by—

$$\frac{S\left(\frac{\pi}{2}\right)}{S(0)}=\frac{\frac{1}{\pi^2}+\frac{2e}{9}-\frac{7e^2}{45}}{\frac{1}{\pi^2}-\frac{2e}{9}-\frac{7e^2}{45}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The value of this ratio is always greater than 1 (unity).

Experimental programme

The experimental programme was designed to determine (*1*) how good an approximation is obtained by using only the first two terms of the Fourier expansion to describe the scanning distribution, equation (*20*); (*2*) the accuracy with which the angular fibre distribution can be calculated from equation (*21*); (*3*) the agreement between the calculated and measured values of N_c, equation (*25*).

It was impossible to produce oriented 2-D sheets. Instead, 30 g/m^2 sheets, composed of a 5:1 mixture of undyed, highly beaten, bleached sulphite fibres and unbeaten sulphate fibres dyed black, were made by lifting a sheet-machine gridplate out of a dilute pulp suspension at an angle. Comparable random sheets were formed from the mixture in the sheetmachine by the standard pulp evaluation procedure. In photographs of the sheets, the black fibres appeared to form a 2-D network, the white fibres being hardly visible (Fig. 1*c*).

The oriented sheets were scanned every 10°, beginning at the apparent machine-direction. The dotted line shown in Fig. 9 is the theoretical curve of the scanning distribution, equation (*20*), having calculated $a=156$ (mean number of intersections per scan), $b_1=38{\cdot}3$, $\alpha=7°$. The experimental points, denoted by crosses, are seen to fall reasonably close to the theoretical curve of eccentricity 0·208, as calculated from equation (*22*). The solid line in

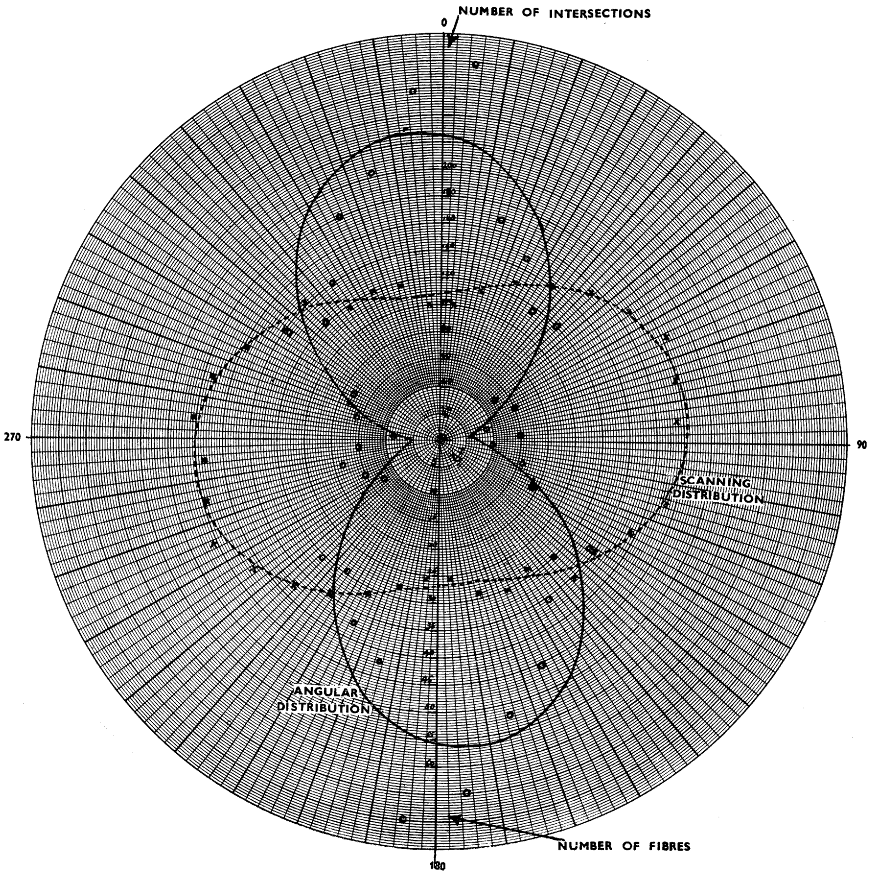


Fig. 9—Scanning distribution and distribution of fibre orientation

Fig. 9 is the angular distribution curve of equation (*21*). The small circles about it, signifying the number of fibres at a given orientation, scatter more than the experimental points of the scanning curve, probably because the number of fibres per 10° sector averages about a quarter of the number of intersections per scan.

The lack of constriction in the scanning curve indicates the insensitivity of scanning to fibre orientation. Real deviations from the cycloidal shape as reported in the literature[6] will not be revealed by an equation as simple as equation (*20*). The Fourier series may be expanded, however, to a sufficiently large number of terms, $N(\theta)=a+b_1 \cos 2\theta+b_2 \cos 4\theta+\ldots+b_n \cos 2n\theta$, so that any desired accuracy can be achieved, with proportionate increase in numerical work.

TABLE 1—SCANNING DATA FOR A RANDOM AND AN ORIENTED 2-D SHEET

Data	*Random*	*Oriented*
Mean fibre length, cm	0·283	
Mean curl factor	1·21	
Area of sheet, cm^2	5·86	
Total scan length, cm	7·5	
Eccentricity	0	0·208
Number of intersections (total scan length)	230	156 (mean)
Number of fibres (calculated)	1 210	818
Number of crossings (calculated)	4 330	1 780
Number of crossings, eccentricity not considered	—	1 910
Counted number of crossings	4 103	1 661

The number of fibres and the number of crossings were counted in both oriented and random sheets, the latter having a nearly circular angular orientation distribution. The data in Table 1 show good agreement between theory and experiment. On the whole, the effect of non-random fibre orientation on the total number of crossings appears to be comparatively small. It will, in general, equal $165e^2$ per cent, which, for the case of the relatively biased orientation illustrated by Fig. 9, amounts to a modification of only 7 per cent.

Flocculation in a 2-D network

THE difficult problem of describing quantitatively the statistical geometry of a flocculated fibre network has not been solved completely. A method that appears promising combines two models, one containing a detailed definition of a floc, the other a detailed definition of a non-random point distribution. In combination, the models give the total number of fibre crossings and, therefore, all the other dependent variables.

The first model defines a floc of density n as a set of n fibres all or most of which intersect each other. The floc density n has a distribution with moments

μ_r. It can be shown that the total number of crossings in a flocculated fibre network is the sum of the number of crossings within flocs plus the number between flocs—

$$N_c = \frac{1}{2} N_f \frac{\mu_2 - \mu_1}{\mu_1} + \frac{1}{2} N_f^2 P \quad \ldots \ldots \quad (27)$$

The term P is the probability that any two fibres, chosen at random, will intersect.

Flocs may constitute only a part of the network, the rest consisting of randomly distributed fibres. In flocculated networks, the probability that a given pair of fibres intersects is a combination of three terms—*(1)* is $2\lambda^2/\pi\bar{r}^2 A_{\text{random}}$ for the random part of the sheet, *(2)* is ~ 1 for intersections within the flocs and *(3)* is $(a_i + a_j + U_i U_j/2)/A_{\text{flocculated}}$[7] for intersections between flocs, where a denotes the area and U the circumference of the flocs.[4]

In order to evaluate equation *(27)*, the area distribution and at least the first two moments of the density distribution of the flocs must be known. These two quantities are those most workers in the field believe to be sufficient for a numerical description of sheet formation.

The second of the two models comes in when we wish to evaluate the area and density distribution of the flocs. Consider circles, denoting flocs, dropped at random on a plane. Both the plane and the circles contain points that may represent either fibre centres or crossings. The background density of points is λ_0, the density within a circle is λ_1 and a region where r circles' overlap is $r\lambda_1$. Then it can be shown that $p(n)$, the number of points per unit area (local point density) in a randomly chosen region of the area A[8], is (*see* Appendix 2)—

$$p(n) = e^{-\alpha}\left[\frac{\lambda_0^n}{n!} e^{-\lambda_0} + \frac{\lambda_1^n}{n!} \sum_{r=1}^{\infty} (\alpha e^{-\lambda_1})^r r^n / r!\right] \quad \ldots \quad (28)$$

In equation *(28)*, r represents the number of overlapping flocs and α, which accounts for the area covered by the flocs, is equal to $\sum \pi \rho_i^2 \delta_i$; the term δ_i is the mean density of centres of circles of radius ρ_i. Rather than attempt to describe the cumbersome evaluation of equation *(28)*, we present below a simplified case to demonstrate its use and the type of information it yields.

Consider the case of non-overlapping flocs of uniform area: that is, $r = 1$, $\alpha = \pi\rho^2\delta =$ fraction of sheet area covered by flocs and α is small enough that $e^{-\alpha} \simeq 1 - \alpha$. Hence, equation *(28)* reduces to—

$$p(n) = (1-\alpha) \frac{\lambda_0^n}{n!} e^{-\lambda_0} + \alpha \frac{\lambda_1^n}{n!} e^{-\lambda_1} \quad \ldots \ldots \quad (29)$$

Equation (*29*) contains three variables, α, λ_0 and λ_1, which terms can be computed by solving equation (*29*) for three sets of data. The data are obtained by dividing A into a large number of small unit areas and counting the number of points per unit area.

To use equation (*29*) for $n=0$, 1 and 2, however, the sheet must be divided into areas so small that they contain only 0, 1 and 2 points. This being impracticable, the range of point densities was divided into three classes covering most of the range, but not entirely, to ensure independence. Equation (*29*) then becomes—

$$f(m_i) = (1-\alpha) \sum_{n(m_i)} \frac{\lambda_0{}^n e^{-\lambda_0}}{n!} + \alpha \sum_{n(m_i)} \frac{\lambda_1{}^n e^{-\lambda_i}}{n!} \quad (i = 1, 2, 3) \quad . \quad (30)$$

where the summation is made over the class widths with means m_i.

The above method was used to study flocculated sheets like that photographed in Fig. 1*d*. Sheets of this kind were made from a mixture of black and white fibres similar to those used for oriented sheets. Unlike the latter, however, they were formed in a British sheetmachine from suspensions containing decreasing amounts of water (increasing concentrations), hence, with an increasing degree of flocculation.

TABLE 2—DISTRIBUTION OF FIBRE CROSSINGS IN A FLOCCULATED 2-D SHEET

(*unit squares of 2 mm side length*)

22	19	11	21	27	60	31	39	18	11	13	12	28	28	21	28	30	49	35	21	42	11
25	26	16	21	22	33	16	27	18	27	27	19	26	29	16	14	22	9	58	14	4	21
24	27	9	54	28	14	18	11	17	27	32	31	46	21	14	23	28	31	18	18	1	14
14	42	27	25	28	25	27	17	24	39	42	39	36	32	24	27	47	58	54	22	26	18
58	54	16	40	32	26	38	21	24	41	24	18	32	51	45	25	37	12	21	11	18	28
21	12	20	33	27	34	34	28	22	51	68	62	21	30	16	27	30	28	27	10	5	27
45	28	22	33	21	14	37	70	56	31	37	17	9	27	18	40	87	77	38	20	50	25
28	17	27	24	27	41	20	37	43	23	32	33	23	21	10	23	16	39	40	38	43	30
32	17	16	12	29	27	12	39	34	21	26	57	25	18	40	16	29	20	71	17	14	13
19	17	13	17	16	25	31	13	36	24	17	40	40	29	45	32	19	61	63	31	37	13

Total number of crossings	6 232
Total number of unit areas (squares)	220
Mean density of crossings	28
Distribution—	

Class	*Range of crossings*	*Number of squares*	*Frequency*
1	1–20	67	$0{\cdot}305=f_1$
2	21–40	120	$0{\cdot}545=f_2$
3	41–60	24	$0{\cdot}110=f_3$
Excess	> 60	9	0·040

A surface photograph like Fig. 1*d* shows part of a flocculated black 2-D sheet. Its negative was projected on to a screen having a square grid to represent 220 squares of 2 mm side length on the original sheet. The fibre crossings were taken as points and their number counted in each square: Table 2 shows the results. The class widths of Table 2 yield, upon computation, $\lambda_0 = 23$, $\lambda_1 = 42$, $\alpha = 0{\cdot}2$; 20 per cent of the area of the sheet is flocculated and has a mean fibre density 42, whereas the rest of the sheet is a random fibre network with mean density 23. The average point density over the entire sheet is 28.

3-D networks

THE geometric treatment of three-dimensional structures is difficult in comparison with the two-dimensional case, not only because it is more complicated, but also because experimental tests by direct visual observation are practically impossible. For lack of a direct check, we must place confidence in the theoretical results and apply them to reasonably well-defined physical experiments. This will provide an indirect check of the theory, besides accomplishing one of the general objectives of the programme—relation of the physical behaviour of paper to its structure. The latter subject is important enough to occupy an article of its own later in the symposium. Only the principal features of 3-D networks will be set forth here.

Multi-planar network

A multi-planar (MP) sheet consists of a pile of N_L distinct, 2-D sheets. Its most important dependent variable is again its total number of fibre crossings. The detailed treatment of MP sheets[9] first considers N_L 2-D layers, each containing N'_f lines of no width, forming on the average N'_c crossings. (Primed quantities refer to networks whose fibres have no width.) If every layer completely penetrates every other layer, the resulting network of $N_L N'_f$ lines lies in a single plane and has $N'_c N_L{}^2$ crossings. If, on the other hand, crossings are formed only by the lines *within* layers and *between* contiguous layers, a total of $N'_c\,(3N_L - 2)$ crossings will be formed.

The number of crossings in an MP sheet of real fibres, N_{cm}, lies somewhere between these two extremes. It is a much more complex quantity than N_c for 2-D sheets, for it cannot be truly evaluated without first considering the effects of fibre width and rigidity. *Within* 2-D layers, either by themselves or *in* MP sheets, the number of crossings is fewer than that predicted by equation (*5*), because fibre width lowers the probability of a crossing between a given pair of fibres. Quantitatively, its effect is to reduce N'_c by a factor $B = \frac{1}{2}[1 + p(0)]$, where $p(0)$ represents the fractional uncovered area of a 2-D sheet and is equal to exp $(=N_f\bar{\alpha}/A)$, the first term of the Poisson distribution. *Between* a pair of 2-D layers in MP sheets, the reduction factor is B^2.

The number of crossings between fibres of different layers is restricted also by the rigidity (or lack of flexibility) of the fibres. The more flexible the fibres, the greater the *fraction of the possible crossings actually formed between the fibres of any two layers*, an independent variable of fibres we have called the penetration factor, σ_K.* Its magnitude, which for fibres $(K-1)$ layers apart is obviously a decreasing function of K, depends on process variables like pressing and drying, as well as fibre properties. For the present, however, our only interest is in the numerical values of σ_K for finished sheets.

From this discussion, it is apparent that no reasonable expression for N_{cm} can be very hospitable. Notwithstanding, a number of simplifications are possible, particularly that $\sigma_{K\geqslant 3}=Cp(0)^{K-2}$, where C is a constant. Then—

$$N_{cm} = N'_c N_L B\left\{\sigma_1+2B\left[1-\frac{1}{N_L}\right]\left[\sigma_2+\frac{Cp(0)^2}{1-p(0)^2}\right]-\frac{2Cp(0)^2[1-p(0)^{2(N_L-1)}]}{N_L[1-p(0)^2]^2}\right\} \quad . \quad . \quad . \quad . \quad (31)$$

In practice, the right side of equation (*31*) produces a number that, replacing N_c in equations (*6*), (*7*) and (*10*) for 2-D sheets, makes it possible to calculate the dependent variables $\bar{c}_m$, $\bar{g}_m$ and n_{gm}. (The subscript m indicates properties of MP sheets.) All three have distributions and means different for every layer in the sheet. In the case of $\bar{c}_m$, for example, the mean number of crossings per fibre in the Kth layer, when $2\leqslant K\leqslant N_L-1$, is—

$$\bar{c}_m = \bar{c}'B\left\{\sigma_1+B\left[2\sigma_2+\frac{Cp(0)^2}{1-p(0)^2}\left(2-p(0)^{2(N_L-K-1)}-p(0)^{2(K-1)}\right)\right]\right\} \qquad (32)$$

Note that $\bar{c}'$ is the mean number of crossings per fibre of no width in a single 2-D layer. For the two edge layers, $K=1$ or N_L and—

$$\bar{c}_1 = \bar{c}'B\left\{\sigma_1+B\left[\sigma_2+\frac{Cp(0)^2}{1-p(0)^2}\left(1-p(0)^{2(N_L-2)}\right)\right]\right\} \quad . \quad (32a)$$

Computation with equations (*32*) and (*32a*) showed the ratio of $\bar{c}_1:c_{k>3}$ to be approximately 1 : 1·5—that is, fibres in the edge layers are held to the network by about two thirds as many crossings as fibres well within the sheet. Consequently, the edge layers should be more porous than the rest of the sheet.

Near the beginning of this article, we intimated that certain problems concerned with MP theory are of a fundamental nature. Evaluation of the dependent variables, N_h, a_h and n_s, for multi-planar sheets is one of those problems. We have not been able to give a satisfactory definition of a *pore* or

* The penetration factor has been called 'the effective fibre flexibility'[9]

a *hole* in an MP network in which the fibres lie in adjacent parallel planes, let alone in truly three-dimensional networks. We have, instead, formulated a model from an MP sheet based on flow phenomena. This is discussed in our second contribution to this symposium.

Experimental programme and results

Our experimental work with MP sheets consisted of measuring the properties of the fibres, particularly the σ_K values and the macroscopic physical properties of MP and normally formed handsheets of the same basis weight. Again, we save discussion of the latter until later in the symposium.

As for the σ_K values, they were measured on three-layer sheets, of which the fibres of the top layer were undyed, those of the middle layer were dyed with Congo Red and those of the bottom layer were dyed with Methyl Violet. Examining these sheets under polarised, vertical illumination according to the technique of Page and Tydeman,[10, 11] the fraction of the area of fibre/fibre crossings in optical contact of almost all crossings could be measured. Typical results appear in Table 3. Greater in sulphite sheets than in kraft sheets, σ_K is increased in both by heavier wet pressing. The accompanying data on the tensile strength of sheets of the two pulps indirectly verifies the evidence of a larger amount of bonding in the sulphite handsheets.

TABLE 3

Pulp	*Pressing pressure, lb/in²*	σ_1	σ_2	σ_3 *at* $p(0)$	*Tensile strength per cm per g,* cm × 10⁻⁵*
Sulphite	1 000	0·95	0·89	0·64 at 0·43	6·1
	50	0·92	0·71	0·44 at 0·44	5·6
Kraft	1 000	0·84	0·65	0·37 at 0·48	4·9
	50	0·74	0·51	0·27 at 0·52	4·0

* Based on eight-layer MP sheets

These results, however, together with other experiments in which mechanical properties of MP sheets of the same substance were compared, represent physical properties and will therefore be discussed later.

Network of horizontal fibres

Here, we are concerned with a network of *rigid* fibres of mean length $\bar{\lambda}$, width $\bar{\omega}$ and thickness $\bar{\delta}$, whose centres are randomly distributed throughout a volume $V=AD$. The fibre axes lie in parallel planes, in which they have a random angular distribution. The terms $\bar{\delta}$ and D, the thickness of the fibres and the maximum thickness of the sheet are new independent variables. Our first problem is again calculation of the number of crossings.

The general equation for the total number of crossings is the same as equation (*14*). The probability that two horizontal fibres intersect is the product of (*1*) the probability that the projections of two fibres intersect in a plane A of the network, $P_1 = 2\bar{\lambda}^2/\pi A$ and (*2*) the probability that two lines of length δ intersect (overlap) on a line of length D (the thickness of the network). This probability P_2 is equal to—

$$P_2 = \frac{\bar{\delta}(2D - 3\delta)}{(D-\delta)^2} \quad \quad (33)$$

For thick sheets, when $D \gg \delta$, equation (*33*) reduces to $2\delta/D$.

Substituting equation (*33*) in equation (*14*) gives—

$$N_c = \frac{\bar{\lambda}^2 N_f^{\ 2} \bar{\delta}(2D - 3\bar{\delta})}{\pi A (D - \bar{\delta})^2} \quad \quad (34)$$

For thick sheets—

$$N_c = \frac{2\bar{\lambda}^2 \bar{\delta} N_f^{\ 2}}{\pi V} \quad \quad (35)$$

Excepting N_h and a_h, the other dependent variables and their distributions can be computed from these equations. A special feature of this network is that its surface profile and variations in its thickness can be calculated.

Real fibres are not truly rigid, however; they are deformed and in contact with more fibres than the above equations indicate. Our efforts to introduce a quantity equivalent to the penetration factor σ_K have so far been unsuccessful.

Squeezed-out 3-D random network

The geometry of a truly 3-D fibre network squeezed out until the angles between the fibres and plane of the sheet are relatively small has so far proved to be too difficult to be treated rigorously. Little enough is known about truly 3-D networks. Miles[(4)] has shown that the total number of crossings made by N_f fibres of length λ and radius r randomly distributed in a volume V (where $V \gg \lambda^3$) is—

$$N_c = \frac{\pi}{2} N_f^{\ 2} \frac{\bar{\lambda}^2 r}{V} \quad \quad (34)$$

Ogston[(12)] has considered the voids for the same model.

Conclusion

A rigorous mathematical treatment of fibre networks to give an exact geometric description of the structure of paper would seem to lie beyond our grasp in the light of present knowledge. Through continued research we hope to arrive at an approximation, although there can be little doubt by now that even this goal is some distance away. Whenever models have to be substituted for reality, physical experiment is the only guide for rightness of choice. It is the physical and, in the end, technical application that justifies any attempt to reach a quantitative understanding of the geometric structure of paper.

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Appendix 1

Orientation

A. SCANNING AND ANGULAR DISTRIBUTION

CONSIDER a scan line of length L in an area A making an angle ϕ with the fixed direction (Ox). The probability that a fibre of length λ dropped randomly at an angle θ on to A is intersected by the scan line at the angle $(\theta-\phi)$ is—

$$\frac{L\lambda|\sin(\theta-\phi)|}{A}, \quad \text{see equation } (19a).$$

The probability that they intersect at any angle is—

$$\frac{L\lambda}{A}\int_0^\pi |\sin(\theta-\phi)|\,\Theta|\theta|\,d\theta$$

and the number of intersections made by the scan line with N_f fibres is—

$$\begin{aligned} N(\phi) &= \frac{N_f\lambda L}{A}\int_0^\pi |\sin(\theta-\phi)|\,\Theta(\theta)\,d\theta \\ &= \frac{N_f L\lambda}{A}\left[-\int_0^\phi \sin(\theta-\phi)\,\Theta(\theta)\,d\theta+\int_\phi^\pi \sin(\theta-\phi)\,\Theta(\theta)\,d\theta\right] \quad . \quad . \quad (A1) \end{aligned}$$

Here, $\Theta(\theta)$ is the angular distribution of the fibre axes.

Differentiating equation (*A1*) twice with respect to ϕ gives—

$$\frac{d^2N(\phi)}{d\phi^2} = \frac{N_fL\lambda}{A}\left\{\left[\int_0^{\phi} \sin(\theta-\phi)\,\Theta(\theta)\,d\theta+\Theta(\phi)\right]\right.$$

$$\left.-\left[\int_0^{\pi} \sin(\theta-\phi)\,\Theta(\theta)\,d\theta-\Theta(\phi)\right]\right\} \quad . \quad (A2)$$

Adding equations (*A1*) and (*A2*) and changing the variable ϕ to θ gives—

$$N(\theta)+\frac{d^2N(\theta)}{d\theta^2} = \frac{2N_fL\lambda}{A}\Theta(\theta)$$

or

$$\Theta(\theta) = \frac{A}{2N_fL\lambda}\left[N(\theta)+\frac{d^2N(\theta)}{d\theta^2}\right] \quad . \quad . \quad . \quad . \quad . \quad (A3)$$

By definition—

$$\int_0^{\pi}\Theta(\theta)\,d\theta = 1.$$

In order to obtain a parallel relation for $N(\phi)$, we integrate equation (*A1*) with respect to ϕ from $\phi=\theta$ to $\phi=\theta+\pi$. Thus—

$$\begin{aligned}\int_{\theta\,0}^{\pi} N(\phi)\,d\phi &= \int_{0\,\cancel{\theta}}^{\theta+\pi} N(\phi)\,d\phi \\ &= \frac{N_fL\lambda}{A}\int_{\theta}^{\theta+\pi}\left[\int_0^{\pi}|\sin(\theta-\phi)|\,\Theta(\theta)\,d\theta\right]d\phi \\ &= \frac{N_fL\lambda}{A}\int_0^{\pi}\left[\int_{\theta}^{\theta+\pi}|\sin(\theta-\phi)|\,d\phi\right]\Theta(\theta)\,d\theta \\ &= \frac{N_fL\lambda}{A}\int_0^{\pi}2\Theta(\theta)\,d\theta \\ &= \frac{2N_fL\lambda}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A4)\end{aligned}$$

Combining equations (*A3*) and (*A4*) results in—

$$\Theta(\theta) = \frac{\dfrac{d^2N(\theta)}{d\theta^2}+N(\theta)}{\displaystyle\int_0^{\pi} N(\phi)\,d\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A5)$$

We now introduce $N(\theta)=a+b\cos 2\theta$, equation (*20*) into equation (*A5*). This produces equation (*21*)—

$$\Theta(\theta) = \frac{1}{\pi}+e\cos 2\theta$$

where $e=-3b/\pi a$, equation (*22*).

B. METHOD OF LEAST SQUARES

The chosen machine-direction ($\theta=0$) may differ by an angle α from the direction of maximum orientation. To find α, we rewrite equation (*20*) as follows—

$$N(\theta) = a+b\cos 2(\theta+\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad (20a)$$

The problem is to solve for a, b and α from a number of scans in different directions. Introducing—

$$\cos(2\theta+2\alpha) = \cos 2\theta \cos 2\alpha - \sin 2\theta \sin 2\alpha,$$
$$N(\theta) = a+m\cos 2\theta + p\sin 2\theta$$

where

$$\left.\begin{aligned} m &= b\cos 2\alpha \\ p &= -b\sin 2\alpha \end{aligned}\right\} \text{ Therefore, } \left.\begin{aligned} b &= \sqrt{m^2+p^2} \\ \alpha &= \tfrac{1}{2}\tan^{-1}(-p/m) \end{aligned}\right\} \textit{ see } \text{equation } (23)$$

According to the method of least squares—

$$\sum [N(\theta)-a-m\cos 2\theta - p\sin 2\theta]^2 = \text{minimum} \quad . \quad . \quad (A6)$$

Expanding equation (*A6*) and remembering—

$$\sum \cos^2 2\theta = \sum \sin^2 2\theta; \qquad \sum \cos 2\theta \sin 2\theta = 0,$$

one obtains

$$\sum N^2(\theta) - 2\sum (a+m\cos 2\theta+p\sin 2\theta)N(\theta) + (m^2+p^2)\sum \cos^2 2\theta + na^2 = \text{minimum},$$

where n is the number of scans (directions), ranging from 0 to π.

Differentiating with respect to a, m and p, gives finally—

$$\left.\begin{aligned} \frac{\partial}{\partial a} &= na - \sum N(\theta) = 0 & \therefore\ a &= \sum N(\theta)/n; \\ \frac{\partial}{\partial m} &= m\sum \cos^2 2\theta - \sum N(\theta)\cos 2\theta = 0 & \therefore\ m &= \frac{\sum N(\theta)\cos 2\theta}{\sum \cos^2 2\theta} \\ \frac{\partial}{\partial p} &= p\sum \cos^2 2\theta - \sum N(\theta)\sin 2\theta = 0 & \therefore\ p &= \frac{\sum N(\theta)\sin 2\theta}{\sum \sin^2 2\theta} \end{aligned}\right\} . \quad (23)$$

When the direction of maximum orientation is known ($\theta=\alpha=0$), the eccentricity e can be calculated from scans in the two main directions ($\theta=0$, $\theta=\pi/2$). Introducing equation (*21*) into equation (*A1*) and changing the variable—

$$N(\theta) = \frac{N_f L\lambda}{A}\left[-\int_0^\theta \sin(\phi-\theta)\left(\frac{1}{\pi}+e\cos 2\phi\right)d\phi + \int_\theta^\pi \sin(\phi-\theta)\left(\frac{1}{\pi}+e\cos 2\phi\right)d\phi\right].$$

Comparing with equation (*20*)—

$$a = \frac{N_f L\lambda}{\pi A}\left[-\int_0^\theta \sin(\phi-\theta)\,d\phi + \int_\theta^\pi \sin(\phi-\theta)\,d\phi = \frac{2N_f L\lambda}{\pi A}.\right.$$

From equation (*22*)—

$$b = -\frac{\pi a e}{3} = -\frac{2eN_f L\lambda}{3A}.$$

Thus, equation (*20*) becomes—

$$N(\theta) = \frac{2N_f L\lambda}{A}\left(\frac{1}{\pi}-\frac{e}{3}\cos 2\theta\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (A7)$$

For $\theta=0$ and $\theta=\pi/2$, respectively—

$$N(0) = \frac{2N_f L\lambda}{A}\left(\frac{1}{\pi}-\frac{e}{3}\right); \qquad N\left(\frac{\pi}{2}\right) = \frac{2N_f L\lambda}{A}\left(\frac{1}{\pi}+\frac{e}{3}\right)$$

Hence—

$$\frac{N(\pi/2)+N(0)}{N(\pi/2)-N(0)} = \frac{2/\pi}{2e/3} = \frac{3}{\pi e}$$

and

$$e = \frac{3}{\pi}\frac{N(\pi/2)-N(0)}{N(\pi/2)+N(0)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Comparing the above value for a with that from equation (*23*)—

$$\frac{\sum N(\theta)}{n} = \frac{2N_f L\lambda}{\pi A}$$

or

$$N_f = \frac{\pi A}{2\lambda}\frac{\sum[N(\theta)/L]}{n} = \frac{\pi A}{2\lambda}\left(\frac{\bar{N}}{L}\right).$$

This result means that the number of fibres in an oriented sheet can be calculated using the scanning technique in the same way as in the case of a random sheet, equation (*16*), if the scanning is done in a number of different directions and the average ratio used in the scanning equation.

C. TOTAL NUMBER OF FIBRE CROSSINGS

Introducing equation (*21*) into equation (*19*)—

$$P = \frac{\lambda^2}{A}\int_0^\pi\int_0^\pi |\sin(\theta-\phi)|\left(\frac{1}{\pi}+e\cos 2\theta\right)\left(\frac{1}{\pi}+e\cos 2\phi\right)d\phi d\theta.$$

For symmetry reasons—

$$\frac{AP}{2\lambda^2} = \int_0^\pi\int_0^\theta \sin(\theta-\phi)\left(\frac{1}{\pi}+e\cos 2\theta\right)\left(\frac{1}{\pi}+e\cos 2\phi\right)d\phi d\theta.$$

The integration is done as follows—

$$\frac{AP}{2\lambda^2} = \int_{\theta=0}^{\theta=\pi}\left(\frac{1}{\pi}+e\cos 2\theta\right)\left[\frac{1}{\pi}\int_{\phi=0}^{\phi=\theta}\sin(\theta-\phi)\,d\phi+e\int_{\phi=0}^{\phi=\theta}\sin(\theta-\phi)\cos 2\phi\,d\phi\right]d\theta.$$

Substituting $\sin(\theta-\phi) = \sin\theta\cos\phi-\sin\phi\cos\theta$

and $\cos 2\phi = 1-2\sin^2\phi = 2\cos^2\phi-1$ gives—

$$\frac{AP}{2\lambda^2} = \int_{\theta=0}^{\theta=\pi}\left(\frac{1}{\pi}+e\cos 2\theta\right)\left\{\frac{1}{\pi}\Big[\sin\theta\sin\phi+\cos\theta\cos\phi\Big]_{\phi=0}^{\phi=\theta}\right.$$
$$\left.+\,e\sin\theta\Big[\sin\phi-\tfrac{2}{3}\sin^3\phi\Big]_{\phi=0}^{\phi=\theta}-e\cos\theta\Big[-\tfrac{2}{3}\cos^3\phi+\cos\phi\Big]_{\phi=0}^{\phi=\theta}\right\}d\theta$$
$$= \int_0^{\pi}\left(\frac{1}{\pi}+e\cos 2\theta\right)\left[\frac{1}{\pi}(1-\cos\theta)+e(\sin^2\theta-\tfrac{2}{3}\sin^4\theta+\tfrac{1}{3}\cos^4\theta-\cos^2\theta\right.$$
$$\left.+\tfrac{1}{3}\cos\theta)\right]d\theta$$
$$= \frac{1}{\pi}-\frac{e^2\pi}{6}.$$

Thus,
$$P = \frac{2\lambda^2}{A}\left(\frac{1}{\pi}-\frac{e^2\pi}{6}\right)$$

and the total number of crossings in the sheet is—

$$N_c = \frac{N_f{}^2\lambda^2}{A\tau^2}\left(\frac{1}{\pi}-\frac{e^2\pi}{6}\right) \quad \ldots\ldots\ldots \quad (25)$$

D. THE TOTAL NUMBER OF CROSSINGS WITH OTHER FIBRES OF THE FIBRES INTERSECTING A SCANNING LINE

Consider a scanning line of length L drawn at an angle ϕ with a fixed direction through a network of fibres with angular distribution $\Theta(\theta)d\theta$. The fibres intersecting the scanning line have an angular distribution—

$$\frac{|\sin(\theta-\phi)|\,\Theta(\theta)\,d\theta}{\int_0^{\pi}|\sin(\theta-\phi)|\,\Theta(\theta)\,d\theta} \quad \ldots\ldots\ldots \quad (A8)$$

This results from the fact that a fibre at an angle θ_1 with the fixed direction is $|\sin(\theta_1-\theta)|/|\sin(\theta_2-\theta)|$ times more likely to intersect the scanning line (making an angle ϕ with the fixed direction) than a fibre making an angle θ_2 with the fixed direction.

From equation (*A1*), it follows that the mean number of intersections with other fibres made by a fibre at an angle θ with the fixed direction is—

$$c(\theta,\lambda) = \frac{N_f\lambda^2}{A}\int_0^{\pi}|\sin(\theta-\psi)|\,\Theta(\psi)\,d\psi \quad \ldots \quad (A9)$$

Then the mean number of crossings per fibre, given the fibre intersects a scanning line that makes an angle ϕ with the fixed direction, equals—

$$T(\phi) = \frac{\int_0^\pi c(\theta, \lambda)|\sin(\theta-\phi)|\,\Theta(\theta)\,d\theta}{\int_0^\pi |\sin(\theta-\phi)|\,\Theta(\theta)\,d\theta} \quad . \quad . \quad . \quad . \quad (A10)$$

and the total number of crossings with other fibres of the fibres intersecting a scanning line of length L is—

$$S(\phi) = T(\phi)N(\phi, L) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A11)$$

We now introduce the following expressions previously used—

$$\Theta(\theta) = \frac{1}{\pi} + e\cos 2\theta, \text{ equation } (21)$$

$$\left.\begin{aligned} c(\theta, \lambda) &= \frac{2N_f\lambda^2}{A}\left(\frac{1}{\pi} - \frac{e}{3}\cos 2\theta\right) \\ N(\phi, L) &= \frac{2N_f L\lambda}{A}\left(\frac{1}{\pi} - \frac{e}{3}\cos 2\theta\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (A7)$$

and calculate $S(\phi)$ for the two main directions (that is, for $\phi=0$ and $\phi=\pi/2$).

For $\phi=0$, the numerator of equation (*A10*) is—

$$\frac{2N_f\lambda^2}{A}\int_{\theta=0}^{\theta=\pi}\left(\frac{1}{\pi} - \frac{e}{3}\cos 2\theta\right)\left(\frac{1}{\pi} + e\cos 2\theta\right)|\sin\theta|\,d\theta.$$

Note that $\cos 2\theta = 2\cos^2\theta - 1$ and $|\sin\theta|d\theta = -d(\cos\theta)$, $(0 \leqslant \theta \leqslant \pi)$.

Thus, the numerator becomes—

$$-\frac{2N_f\lambda^2}{A}\int_{\theta=0}^{\theta=\pi}\left(\frac{1}{\pi} + \frac{e}{3} - \frac{2e}{3}\cos^2\theta\right)\left(\frac{1}{\pi} - e + 2e\cos^2\theta\right)d(\cos\theta),$$

which on integration gives—

$$\frac{2N_f\lambda^2}{A}\cdot 2\left(\frac{1}{\pi^2} - \frac{2e}{9\pi} - \frac{7e^2}{45}\right).$$

Similarly, the denominator of equation (*A10*) for $\phi=0$ is equal to $2\left(\frac{1}{\pi} - \frac{e}{3}\right)$.

Thus—

$$T(0) = \frac{2N_f\lambda^2}{A}\cdot\frac{\left(\frac{1}{\pi^2} - \frac{2e}{9\pi} - \frac{7e^2}{45}\right)}{\left(\frac{1}{\pi} - \frac{e}{3}\right)}$$

and
$$S(0) = \frac{4N_f{}^2\lambda^3L}{A^2}\left(\frac{1}{\pi^2}-\frac{2e}{9\pi}-\frac{7e^2}{45}\right).$$

Using the same procedure—

$$S\left(\frac{\pi}{2}\right) = \frac{4N_f{}^2\lambda^3L}{A^2}\left(\frac{1}{\pi^2}+\frac{2e}{9}-\frac{7e^2}{45}\right).$$

Hence,
$$\frac{S\left(\frac{\pi}{2}\right)}{S(0)} = \frac{\frac{1}{\pi^2}+\frac{2e}{9\pi}-\frac{7e^2}{45}}{\frac{1}{\pi^2}-\frac{2e}{9\pi}-\frac{7e^2}{45}} > 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Note that, when concerned with tensile strength, ϕ is perpendicular to the direction of the externally applied force.

In deriving equation (*26*), the crossings between fibres intersecting the scanning line were counted twice; those between fibres that do and do not intersect it were counted once. This error being in both the numerator and denominator, the total error in equation (*26*) is relatively small.

Appendix 2

Flocculation—probabilistic treatment of a non-random point distribution[8]

A MODEL to describe a non-random 2-D distribution of points envisages a plane containing randomly placed *circles* of various radii; their centres have a Poisson distribution with densities θ_K. The parts of the plane not covered by circles contain *points* having a Poisson distribution with density λ_0; the parts covered by a circle contain points having a Poisson distribution with higher density, λ_1 ($\lambda_1 > \lambda_0$). The value λ_1 is independent of the radii of the circles. For all cases that r circles overlap, the density is $r\lambda_1$.

The plane is divided into areas of type c_r(c_o, background; c_1, covered by one circle; c_2, covered by two circles, etc.). In any small neighbourhood chosen at random, let N=the local density of points. The probability distribution of N is given by—

$$p(N = n) = p(n)$$
$$= \sum_{r=0}^{\infty} p'(n|c_r)p''(c_r) \quad . \quad . \quad . \quad . \quad . \quad . \quad (A12)$$

where $p'(n|c_r)$ = conditional probability that $N = n$, given that the region is of type c_r

= Poisson distribution with mean λ_0, λ_1, $2\lambda_1$, . . ., $r\lambda_1$.

and $p''(c_r)$ = probability that the region is of type c_r.

= probability that a circle of radius a surrounding the neighbourhood encountered contains r centres

$= e^{-\alpha}\,\alpha^r/r!$

where $\alpha=\pi a^2\sum\theta_K$. When a has a distribution identical to the circle radius distribution, it follows that—

$$\alpha = \sum\pi a_K{}^2\theta_K$$

where θ_K is the mean density of circles of radius a_K.

Thus, equation (*A12*) becomes—

$$\begin{aligned} p(n) &= p'(n|c_0)p''(c_0)+\sum_{r=1}^{\infty} p'(n|c_r)p''(c_r) \\ &= \left(e^{-\lambda_0}\frac{\lambda_0{}^n}{n!}\right)e^{-\alpha}+\sum_{r=1}^{\infty}\left[e^{-r\lambda_1}\frac{(r\lambda_1)^n}{n!}\right]e^{-\alpha}\frac{\alpha^r}{r!} \\ &= e^{-\alpha}\left[\frac{\lambda_0{}^n}{n!}e^{-\lambda_0}+\frac{\lambda_1{}^n}{n!}\sum_{r=1}^{\infty}(\alpha e^{-\lambda_1})^r\frac{r^n}{r!}\right] \qquad (28) \end{aligned}$$

For the initial evaluation of equation (*28*), two simplifications are introduced—(*1*) the circles have uniform radius a so that α is the fractional area of the plane covered by circles, (*2*) there are so few circles that overlapping can be neglected, hence the density distribution reduces to $r=0$, 1 and α is so small that $e^{-\alpha}\approx 1-\alpha$.

Then equation (*28*) becomes—

$$p(n) = (1-\alpha)\frac{\lambda_0{}^r}{n!}e^{-\lambda_0}+\alpha\frac{\lambda_1{}^n}{n!}e^{-\lambda_1} \qquad (29)$$

Equation (*29*) is a distribution containing three parameters λ_0, λ_1 and α, which can be evaluated from three independent sets of data. The analysis is least laborious when the sample areas are so small that they contain no more than four points and f_0, f_1, f_2 and f_3 denote the frequencies of sampling areas containing respectively $n=0$, 1, 2, 3 points. From equation (*29*)—

$$\left.\begin{aligned} f(0) &= (1-\alpha)e^{-\lambda_0}+\alpha e^{-\lambda_1} \\ f(1) &= (1-\alpha)\lambda_0 e^{-\lambda_0}+\alpha\lambda_1 e^{-\lambda_1} \\ f(2) &= (1-\alpha)\frac{\lambda_0{}^2}{2}e^{-\lambda_0}+\alpha\frac{\lambda_1{}^2}{2}e^{-\lambda_1} \end{aligned}\right\} \qquad (A13)$$

Selecting areas containing fewer than 4 points being impracticable in the case of crossings in a 2-D sheet, the range of point densities can be divided into three classes covering most of the range, but not entirely, to ensure independence. Then the individual Poisson frequencies in equation (*29*) must be summed up over the n values in each class width. In the case described in the text, the three classes ranged 1–20, 21–40 and 41–60. Thus, from equation (*29*)—

$$\left.\begin{aligned} f_1(n = 1, 2, \ldots, 20) &= (1-\alpha)\sum_{n=1}^{20} \frac{\lambda_0{}^n}{n!} e^{-\lambda_0} + \alpha \sum_{n=1}^{20} \frac{\lambda_1{}^n}{n!} e^{-\lambda_1} \\ f_2(n = 21, 22, \ldots, 40) &= (1-\alpha)\sum_{n=21}^{40} \frac{\lambda_0{}^n}{n!} e^{-\lambda_0} + \alpha \sum_{n=21}^{40} \frac{\lambda_1{}^n}{n!} e^{-\lambda_1} \\ f_3(n = 41, 42, \ldots, 60) &= (1-\alpha)\sum_{n=41}^{60} \frac{\lambda_0{}^n}{n!} e^{-\lambda_0} + \alpha \sum_{n=41}^{60} \frac{\lambda_1{}^n}{n!} e^{-\lambda_1} \end{aligned}\right\} \qquad (29a)$$

which are the expanded form of equation (*30*).

Transcription of Discussion

DISCUSSION

PROF. R. H. PETERS: Could you comment on how easy it is to make a random sheet and, secondly, can you possibly explain at this stage how it is that the Poisson distribution has come into the picture as distinct from a normal distribution?

DR. H. CORTE: On the first question, when you make a $2\frac{1}{2}$ g/m^2 handsheet, you suspend 50 mg of fibres in 7 litres of water and you will see in the suspension (or you have at least the impression) that the fibres are moving quite independently of one another. We do not know whether they do that. When fibres move independently of one another, they should form a random network: that is the definition of it. Whether or not they do this we do not know beforehand, but we can check it afterwards.

Once this sheet is made, we can perform a number of quite simple experiments. These sheets have a diameter of 16 cm and, for instance, you draw a straight line and count the number of intersections between the straight line and the fibres—of the order of 1 200. Then you divide the straight line into a number of equal sections and compare the number of intersections per section with the Poisson distribution. This has to fit and that answers your second question. When you travel along a straight line and place events at random, the number of events at uniform intervals has a Poisson distribution. Thus, the number of telephone calls in a busy telephone exchange per minute has a Poisson distribution.

The existence of such a Poisson distribution is not a complete proof, but it is very strong support that the system we have produced is random.

MR. P. H. PRIOR: If you perform exactly the same operation with one of your flocculated sheets, how will that differ from the Poisson distribution? Will it necessarily be distinguishable?

DR. CORTE: Yes, it would be distinguishable and it will not be a Poisson distribution. You may have a bimodal distribution, for instance. If not, you may have a wider distribution and that is what normally happens.

DR. A. B. TRUMAN: As stated in your paper, that the fibres in a two-dimensional sheet are randomly distributed does not imply that they are uniformly distributed. Does it follow that flocculation of the fibres may be observed in a random sheet?

DR. CORTE: The term flocculation is, so to speak, a human expression and not a mathematical one. Papermakers use this word for a visual effect: when we use the word here, we mean the amount of non-uniformity that is beyond that inherent in a Poisson process. You can think of making the non-uniformity of a random sheet the standard (unit flocculation). The flocculation scale would then start at this point (the non-uniformity of a random sheet) and the term *flocculated* would apply to sheets with a formation worse than random.

MR. P. E. WRIST: I wish to refer to a point to be discussed in my paper at the end of the week. For reasons outlined there, we have been forced to the conclusion that the fibres in a sheet of paper are more uniformly distributed than would occur by a random distribution alone.

DR. CORTE: I am glad to hear this. I was under the impression that, in order to render a sheet, say, completely uniform, you have to control the deposition of every fibre. Only then can you control the position of the fibre centres and make a completely ordered and uniform structure like a woven fabric.

MR. WRIST: This is the ultimate in control. While we cannot control fibre deposition to any extent approximating to this ideal, we do have a small measure of control on the papermachine through the use of such means as wire shake, velocity differential between the jet and wire speed and the agitation that occurs over table rolls at high speeds—all of which produce relative motion between the fibre suspension and the forming web. Combined with the local variations of drainage resistance produced by any local variations in the concentration of fibre deposition, this relative motion promotes a more uniform distribution than would be produced by a randomising process alone.

DR. CORTE: Relative motion of fibres leading to a more uniform distribution may exist, but I have never seen them and they have not been considered.

MR. W. H. HALE: Have these statistics been compared with actual papers made on standard production machines?

DR. CORTE: It has never been done, because no two-dimensional sheet can be made on the papermachine and this statistical treatment refers to two-dimensional papers only. The extension of statistical geometry to three dimensions is possible and has been made, but direct observation of the geometric quantities is impossible; instead, one has to use them to predict physical behaviour. That will be the subject of our next contribution.

PROF. A. H. NISSAN: I am interested in comparing this work with other work, not in the paper industry at all, but where statistical geometry is called for. Bernal published about two years ago an article in *Nature*, in which he tried to explain (very tentatively as admitted by him) the structure of liquids and he calls for statistical geometry as a new science that does not exist as yet, for a higher form of mathematics. He has done some empirical experiments like yours to find out the number of sides of polygons that are produced randomly and he gives a table of results. I was interested to find that the maximum occurs at five in his work, whereas in yours I notice four sides.

DR. CORTE: The number of sides of the polygons are derivable.

PROF. NISSAN: His were not and I was wondering whether you have compared your work with Bernal's.

DR. CORTE: I know Bernal's publication on crystal lattices, but as a matter of fact the number of sides of the polygons in two dimensions is four: there is strict proof of it, very easy and quoted in our paper. We have in fact made enlarged photographs of such a piece of paper, cut these polygons out with a pair of scissors and written down the number of sides. We found to our surprise that 88 per cent of them had three or four sides, with an average of four. Unfortunately, we did not isolate the pentagons. (Bernal's polygons are the faces of polyhedra, which he finds on the average to be pentagons.)

PROF. NISSAN: So really your results differ from Bernal's.

DR. O. J. KALLMES: I think he is working only in three dimensions, not in two.

DR. C. W. CARROLL: Arising from Wrist's comments, if by *uniformity* he means *symmetry*, I would recall to you that the Poisson distribution can be approximated by the normal distribution under certain conditions. In the case of random fibre deposition, these conditions correspond to the requirement that in spite of the low probability of an event (that is, small chance of a fibre centre landing on a particular small sub-area), the number of fibres involved is so large that the product of the small probability of an event and this very large number of randomly deposited fibres (the parameter of the Poisson distribution) is itself relatively large. Thus, the more concentrated the fibre suspension from which a sheet is randomly formed, the greater the tendency that the sheet will be characterised by a normal distribution of fibre centres. In this sense, the Poisson and the normal distributions are not two discrete distributions, but represent a continuum in the realm of distributions, merging one into the other.

MR. WRIST: My comment did not refer to the choice of distribution function, but that the distributions we obtained on practical sheets of paper are more regular than would result from a random distribution alone. If we could do no better than achieve a random distribution as you seem to suggest and that flocculation worsens the situation still further, we would be unable to make a saleable sheet of paper.

DR. KALLMES: In any case, if you say basically the process gives random flocculation of unity by definition, to a more uniform sheet we could give that a flocculation figure of 0·9 or 0·8. The important point is that you have a reference mark.

A DELEGATE: I am getting a bit confused. Don't you get uniformity from averaging a large number of random distributions?

DR. CORTE: When the mean goes up, the variance goes up accordingly and the standard deviation (the square root) goes up too. The relative standard deviation (standard deviation divided by the mean) goes down with increasing thickness of the pile of random sheets.

MR. WRIST: If we restrict our discussion to two-dimensional sheets alone, a random distribution is acceptable; but, if you then go on to build up a three-dimensional sheet by stacking two-dimensional sheets, you are assuming that the relative position of the fibre in a given layer is completely independent of the positions of fibres in the layers beneath. This is where my interpretation of the papermaking process disagrees with yours. Once you have the first two-dimensional sheet laid down, the deposition of the subsequent fibres is not a completely random event, there is a strong tendency for it to be drawn to a place in the sheet where there is deficiency in fibres and the result is a tendency to build up a much more uniform sheet. It is fortunate that this can occur; otherwise, thin sheets of paper would be completely unacceptable in formation.

DR. CORTE: Nevertheless, although this may be so, it could be checked whether randomness occurs. For machine-made papers, it may not be the case, because the hydrodynamic effect could upset the whole picture. We do not seem to know exactly how the fibres are deposited on the wire. For a purely formal description, it would not really matter, because you could slice a paper and describe each layer no matter how it was formed. We want only to describe, we do not want to refer to the forming process at all. If it is not random, we have experimental means to find out how non-random it is and express this. The problem of how this state of affairs was produced is an entirely different thing and is not the subject matter of this paper.

MR. P. A. TYDEMAN: Could you clarify a point on nomenclature? You have defined g as mean free fibre length and I notice you call it also the distance between two intersections.

DR. CORTE: I mean the distance between centres of fibre intersections.

MR. TYDEMAN: That is surely not free fibre length?

DR. CORTE: We call it free fibre length. The distribution is, by the way, independent of the width of the fibres. When you take wider fibres—we assume that we have a large number of them with statistical or random distribution—then, of course, a number of these gaps would disappear, would be blocked, but those remaining are still an infinitely large number and their distribution would still be exactly the same.

MR. P. G. SUSSMAN: Have you ever used a scanning area smaller than 1 mm and so determined the statistical distribution of the mass of these small areas? If the scanning area is small enough, there is a definite probability of finding very dense spots in a sheet of purely random structure.

I once made sheets of 70 g/m^2 substance from highly dilute stock (0·002 per cent consistency) and they showed quite a few very dense spots, though they were very even over larger areas.

An ordinary handsheet, made from the same pulp at 0·02 per cent consistency, showed more general variation in look-through, but none of these dense spots.

DR. CORTE: We have never scanned areas smaller than 1 mm^2, but we have scanned larger areas by taking, for instance, four of them together to give a square. There is a certain rule about this, how the parameters of the distribution vary from size to size and, whenever they do not correspond to this, then you have a non-random structure. This is revealed by comparing the results of two adjacent squares or of two squares separated by one or more squares. The autocorrelation between squares that are a certain distance apart would indicate whether the distribution is random.

A random distribution is independent of the size of the squares. Only the parameters of the distribution vary with the size of the squares in a well-defined manner.

A DELEGATE: In practice, when one makes a random sheet does it work out according to the equation—and when one scans a sheet made from a high dilution, do you in fact find this so, even when you are scanning very small areas?

DR. CORTE: Yes, this is part of the equation. Take, for instance, the one-dimensional case analogy. The parameter of the negative exponential distribution would be one over the mean number of intersections per unit length, say, 1 mm: this gives the spread of the distribution. When the intervals have only half the width, say, 0·5 mm, then of course the mean is smaller, the parameter is larger and the spread is automatically larger.