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# JOINT PROBABILITY FUNCTION RELATING FIBRE SEGMENTAL LENGTH AND ORIENTATION

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THE purpose of this contribution is to describe how statistical-geometrical analysis [such as that developed in the paper by Corte and Kallmes, this vol., pp. 13–46] can be extended and applied to the determination of the joint probability density function $p(s, \theta)$, which arises so naturally and importantly in Van den Akker's very interesting theoretical analysis of some of the fundamental mechanical properties of paper exhibited in the elastic regime (this vol., pp. 205–241).

It should be recalled that $p(s, \theta)$ is the probability density function describing the joint distribution of segmental or gap length $s$ and segment or fibre orientation $\theta$. This joint probability density function results from considering the general, total joint distribution function $p(s, \theta, A, I)$ as being reasonably equivalent to the product of two relatively independent joint distribution functions, namely, $p(s, \theta)$ and $p(A, I)$.

Corte and Kallmes have shown (this vol., p. 19) that for 2-D, randomly formed isotropic sheets, the probability of a gap length ($g$ in their nomenclature) having a length between $g$ and $(g+dg)$ is—

$$r(g) = \frac{N}{L} \exp\left(-\frac{N}{L} g\right) dg \quad \quad . \; . \; . \; . \; . \; (1)$$

where $N/L$ is the mean number of fibre intersections per unit length of a scanning line in any direction or per unit length of a class of fibres considered to be aligned in any direction. In terms of Van den Akker's nomenclature (gap length designated by $s$), the probability density of equation (1) is—

$$p(s) = \frac{N}{L} \exp\left(-\frac{N}{L} s\right) \quad \quad . \; . \; . \; . \; . \; . \; . \; (2)$$

Equation (2) is the exponential distribution function, which is a special case of the gamma distribution function. This latter function arises from the application of 'waiting time' analysis to Poisson situations[1] such as the Poisson distribution of fibre centres in a randomly formed sheet.

For the *isotropic case* $N/L \neq (N/L)(\theta)$, that is, $N/L$ is constant and not a function of the scanning angle $\theta$ (or particular angle of fibre orientation). In particular, for the isotropic case, the angular distribution function $p(\theta) = 1/\pi$. Under these circumstances, Corte and Kallmes have shown (this vol., p. 22) that—

$$\frac{N}{L} = \frac{2N_f\bar{\lambda}}{A\bar{\tau}\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $N/L$ is the average number of fibres intersected by a scanning line per unit length of the scanning line, $N_f$ is the number of fibre centres in area $A$ and $\bar{\lambda}/\bar{\tau}$ is the average reduced fibre length.

In the *anisotropic case* of machine-made paper, the angular orientation function $p(\theta)$ is not constant, but is a function symmetrical in the machine-direction and dependent on $\theta$. In this case, $N/L = (N/L)(\theta)$, that is, $N/L$ is a function of $\theta$.

Since $N/L$ depends on $\theta$ for the anisotropic case, equation (*2*) must be written as a *joint* probability density function—

$$p(s, \theta) = \left[\frac{N}{L}(\theta)\right] \exp\left\{-\left[\frac{N}{L}(\theta)\right]s\right\} \quad . \quad . \quad . \quad . \quad (4)$$

In order to evaluate $p(s, \theta)$, it is therefore necessary to determine $[(N/L)(\theta)]$, that is, to determine the functional dependence of $N/L$ on $\theta$. This may be done with the aid of Fig. 1, which shows a fibre $F$–$F'$, which makes an average angle of $\gamma$ with a reference direction, say, the cross-direction. The figure shows also an arbitrary scanning line $S$–$S'$, which makes an angle $\theta$ with the reference direction. Using Fig. 1 and making an analysis (similar to that employed by Corte and Kallmes in obtaining equation (*3*) for the isotropic case) of the number of fibres intersected when the anisotropic orientation function is the general, symmetrical density function $p(\theta)$ gives rise to—

$$\left[\frac{N}{L}(\theta)\right] = \frac{N_f\bar{\lambda}}{A\bar{\tau}} \int_\theta^{\theta+\pi} |\sin(\gamma - \theta)| p(\gamma)\, d\gamma \quad . \quad . \quad . \quad . \quad (5)$$

where $p(\gamma) = p(\theta)|_{\theta\to\gamma}$. It should be noted that the orientation distribution function $[p(\theta) \equiv p(\gamma)]$ appears *independently* in equation (*5*), also that equation (*5*) reduces as it should to equation (*3*) for the isotropic case when $p(\theta)$ is a constant, independent of $\theta$ and equal to $1/\pi$.

Equation (*5*) provides the desired functional interrelationship between

$N/L$ and $\theta$ and, when this equation is substituted into equation (*4*), the following expression for $p(s, \theta)$ results—

$$p(s, \theta) = \left[\frac{N_f\lambda}{A\bar{\tau}} \int_{\theta}^{\theta+\pi} |\sin(\gamma-\theta)|\, p(\gamma)\, d\gamma\right] \times$$
$$\exp\left\{-\left[\frac{N_f\lambda s}{A\bar{\tau}} \int_{\theta}^{\theta+\pi} |\sin(\gamma-\theta)|\, p(\gamma)\, d\gamma\right]\right\} \quad . \quad . \quad . \quad . \quad (6)$$

Equation (*6*) gives the desired joint probability density function relating the frequency of distribution of fibre segmental lengths in the range between $s$ and $(s+ds)$ for fibres oriented between angles $\theta$ and $(\theta+d\theta)$.

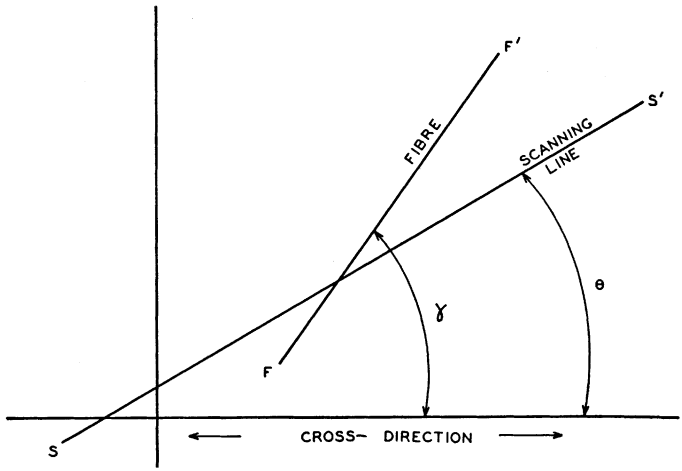


***Fig. 1***

This kind of analysis can be extended to the multi-planar or 3-D case in which the sheet thickness (that is, analysis in the $z$ direction) is important; however, there are several problems involved in making such an extension and these will not be discussed here.

An important point is that through Van den Akker's theoretical analysis, attention has been focused on certain quantities or expressions that have fundamental significance in explaining and predicting the mechanical behaviour of paper and that for at least one of these expressions, namely, $p(s, \theta)$ statistical geometry provides an effective tool for evaluating the desired relationship in explicit, basic terms.

## REFERENCE

1. Parzen, E., *Modern Probability Theory and Its Applications* (John Wiley and Sons Inc., New York, London, 1960)

# DISCUSSION

MR. D. H. PAGE: Dr. Van den Akker has asked whether we have observed bond failure due to torque. Unfortunately, we have not had the opportunity to search our micrographs for this effect, but we have the necessary data recorded for such a search to be made and we will certainly carry this out.

I would like to comment on the question of tension failure at the end bond of a fibre. The picture that Van den Akker has given is, as he obviously realises, a highly idealised one. On average, in a paper sheet each fibre has two bonds on it that are the end bonds of other fibres. As these must have high shear stresses associated with them, it follows that the shear stress distribution on the bonds along the fibre cannot be as he indicates, but is in reality much more complex than this, with quite high shear forces occurring occasionally on bonds remote from the fibre ends.

*Fig. D5*

The prediction that during straining of a sheet failure of bonds along a fibre can occur from one end is borne out, however, by some of our work on bond breakage. We have developed a technique that enables us to reveal the relative movement of fibres on the surface of a sheet during straining. It involves the deposition on the surface of a very thin film of Formvar. During straining, relative movements of the fibres are shown by wrinkles in the film (Fig. D5). The arrow-like effect after straining the sheet was not uncommon in lightly beaten sheets on fibres aligned in the direction of straining and indicates the breakage of the bonds at the end of the fibre and the relative movement of the fibre end with respect to the sheet.

DR. A. B. TRUMAN: The magnitude of the stress that a fibre-to-fibre bond is capable of sustaining, on your estimation, is of great interest to those engaged in research into the surface bonding strength of paper. I think it follows from your results that a stress of several hundred atmospheres might be required to remove fibres from the body of the paper in a direction perpendicular to the plane of the bond. Even on the surface, therefore, the bonds should be capable of withstanding stresses of many atmospheres before being removed. This is very relevant to the results of investigations into the forces of picking in the printing nip. Estimates of the stresses arising in the outgoing nip have so far inclined to values of about 2 atmospheres, but recent work carried out at Manchester College of Science and Technology (to be published shortly) has indicated that the stresses might be much greater than this.

DR. J. A. VAN DEN AKKER: For some time, we have attempted to develop various experimental techniques for the determination of the '*z*-tensile' or 'transverse tensile' strength of paper. Although these techniques have been refined, they are not altogether successful and seem to serve only as guides in experimental programmes. An apparently perfect laboratory test may involve a fundamental difficulty when it is used for measuring fibre-to-fibre bond strength in the *z* direction and I can see no way of evading it—namely, when a sheet is strained in the *z* direction, the deformation of the fibre segments would be expected to result in severe stress concentration at the edges of the bonds, the kind of stress concentration that is involved in peeling. Accordingly, it is expected that the apparent failing stress in the bonds would be much less than the true value, possibly by an order of magnitude.